

Oppgave 1. Løsning.

a) Av symmetrigrunner er

$$\underline{\underline{\langle v_x \rangle = 0}}$$

b) Ekvipartisijonsprinsippet gir

$$\langle \frac{1}{2} m v_x^2 \rangle = \frac{1}{2} kT$$

$$\Rightarrow \underline{\underline{\langle v_x^2 \rangle = \frac{kT}{m}}}$$

c) Av symmetrigrunner er

$$\underline{\underline{\langle v^2 v_x \rangle = 0}}$$

d) Av symmetrigrunner er

$$\underline{\underline{\langle v_x^2 v_y \rangle = 0}}$$

e) Vi har:

$$\begin{aligned} \underline{\underline{\langle (v_x + b v_y)^2 \rangle}} &= \langle v_x^2 \rangle + b^2 \langle v_y^2 \rangle + 2b \langle v_x v_y \rangle \\ &= \frac{kT}{m} + b^2 \frac{kT}{m} + 0 = \underline{\underline{(1+b^2) \frac{kT}{m}}} \end{aligned}$$

Oppgave 2. Løsning

a)

Sannsynlighetsfordelingen for et system med energi E tilstandene m , n

$$p_m = \frac{e^{-\beta E_m}}{\sum_n e^{-\beta E_n}} = \frac{e^{-\beta E_m}}{Z}$$

Middelenergien til systemet er da

$$\begin{aligned} \underline{\underline{U}} = \langle H \rangle &= \sum_m p_m E_m = \frac{1}{Z} \sum_m E_m e^{-\beta E_m} \\ &= -\frac{1}{Z} \frac{\partial}{\partial \beta} \sum_m e^{-\beta E_m} = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = \underline{\underline{-\frac{\partial \ln Z}{\partial \beta}}} \end{aligned}$$

b)

Vi har:

$$\begin{aligned} \underline{\underline{\langle H^2 \rangle}} &= \sum_m E_m^2 p_m = \frac{1}{Z} \sum_m E_m^2 e^{-\beta E_m} \\ &= \frac{1}{Z} \frac{\partial^2}{\partial \beta^2} \left(\sum_m e^{-\beta E_m} \right) = \underline{\underline{\frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}}} \end{aligned}$$

c)

Av a) og b) følger

$$\langle H^2 \rangle - \langle H \rangle^2 = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \left(-\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right)^2$$

$$= \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \frac{1}{Z^2} \left(\frac{\partial Z}{\partial \beta} \right)^2 = \frac{\partial^2}{\partial \beta^2} \ln Z$$

Jelaso

$$\underline{\underline{\Delta U = \sqrt{\frac{\partial^2}{\partial \beta^2} \ln Z}}}$$

Vi han

$$\frac{\partial^2 \ln Z}{\partial \beta^2} = - \frac{\partial}{\partial \beta} \left(- \frac{\partial}{\partial \beta} \ln Z \right)$$

Vho. a) m m di:

$$\underline{\underline{\Delta U = \sqrt{- \frac{\partial U}{\partial \beta}}}} = \underline{\underline{T \sqrt{k \frac{\partial U}{\partial T}}}}$$

d) Varian kapasiteler

$$C_v = \left(\frac{\partial U}{\partial T} \right)_v$$

Alho bin

$$\underline{\underline{\Delta U = T \sqrt{k C_v}}}$$

e)

For en ideell monoatomisk gass er i følge ekvipartisjonsprinsippet

$$U = \frac{3}{2} N k T$$

Da får vi

$$\begin{aligned} \Delta U &= T \sqrt{k \frac{\partial U}{\partial T}} = T \sqrt{k \frac{3}{2} N k} \\ &= k T \sqrt{\frac{3}{2} N} \end{aligned}$$

Det følger

$$\begin{aligned} \frac{\Delta U}{U} &= \frac{k T \sqrt{\frac{3}{2} N}}{\frac{3}{2} N k T} = \frac{1}{\sqrt{\frac{3}{2} N}} \\ &= \underline{\underline{\frac{1}{\sqrt{\frac{3}{2} N}}}} \end{aligned}$$

Oppgave 3. Løsning

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a)

i) For fermioner er $m_k = 0$ eller 1.

$$\Rightarrow \underline{\underline{\langle m_k \rangle}} = \sum_{m_k=0}^1 p(m_k) \cdot m_k = p(1)$$
$$= \frac{e^{\beta(\mu - \epsilon_k)}}{1 + e^{\beta(\mu - \epsilon_k)}} = \underline{\underline{\frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}}} \quad (1)$$

ii) For bosoner er $m_k = 0, 1, 2, \dots$

$$\Rightarrow \sum_{m_k=0}^{\infty} e^{\beta(\mu - \epsilon_k) m_k} = \sum_{m_k=0}^{\infty} \left(e^{\beta(\mu - \epsilon_k)} \right)^{m_k}$$
$$= \frac{1}{1 - e^{\beta(\mu - \epsilon_k)}}$$

Da blir

$$p(m_k) = (1 - e^{\beta(\mu - \epsilon_k)}) e^{\beta(\mu - \epsilon_k) m_k}$$

og

$$\langle m_k \rangle = \sum_{m_k=0}^{\infty} m_k p(m_k) = (1 - e^{\mu}) \sum_{m_k=0}^{\infty} m_k e^{\mu m_k},$$

$$\text{der } \mu = \beta(\mu - \epsilon_k).$$

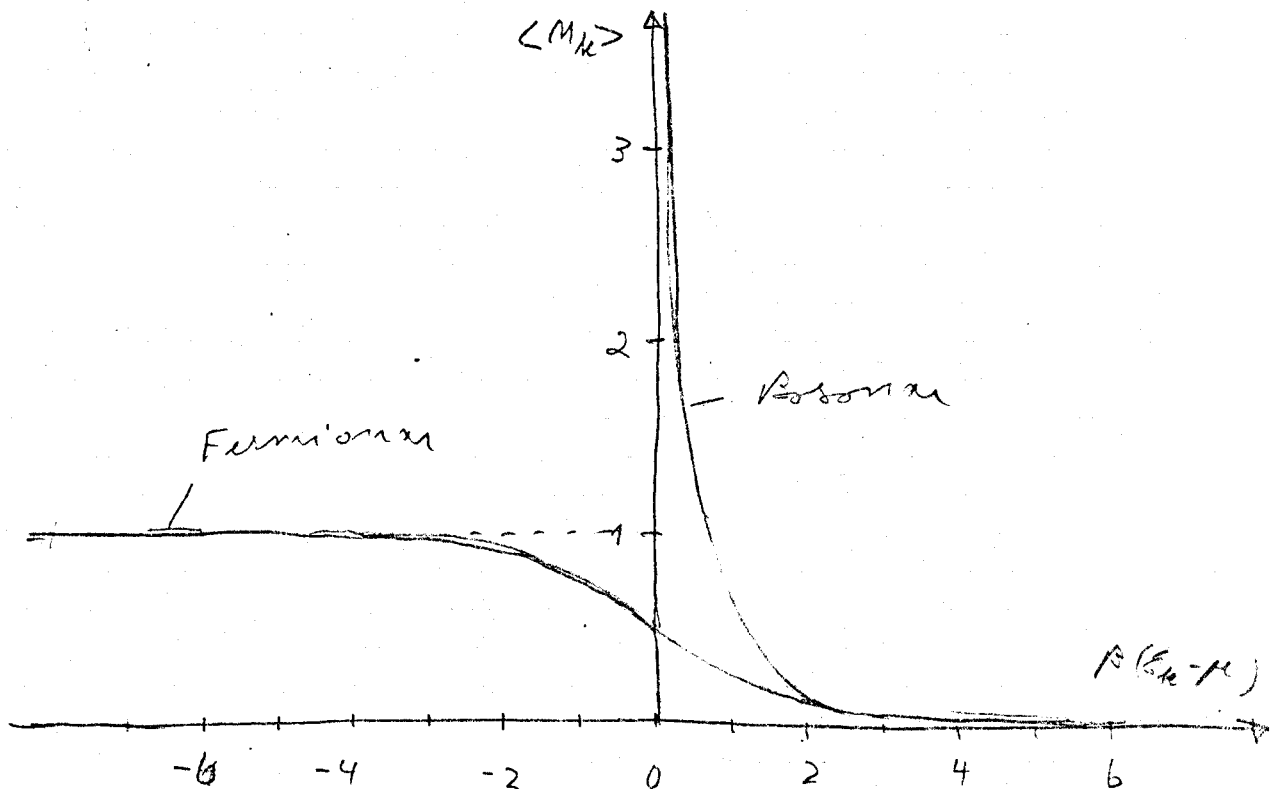
Vi kan deriva

$$\begin{aligned}\langle n_k \rangle &= (1 - e^{-\mu}) \frac{\partial}{\partial \mu} \sum e^{\mu n_k} \\ &= (1 - e^{-\mu}) \frac{\partial}{\partial \mu} \frac{1}{1 - e^{-\mu}} = \frac{e^{-\mu}}{1 - e^{-\mu}}\end{aligned}$$

Alternativt

$$\begin{aligned}\langle n_k \rangle &= \frac{e^{B(\mu - \epsilon_k)}}{1 - e^{B(\mu - \epsilon_k)}} = \frac{1}{e^{B(\epsilon_k - \mu)} - 1} \quad (2) \\ &= \frac{1}{e^{B(\epsilon_k - \mu)} - 1}\end{aligned}$$

b)



NB! For bosons $\mu < \epsilon_k$ for all k . I
motsatt fall for en negativ
bevakning, vilket inte är meningsfullt

c) ...

Vierhan:

$$\langle M_k^2 \rangle = \sum_{M_k} M_k^2 p(M_k)$$

i) Fermionen

$$\langle M_k^2 \rangle = \sum_{M_k=0}^1 M_k^2 p(M_k) = p(1) \stackrel{(1)}{=} \langle M_k \rangle$$

Also: in

$$\underline{\underline{\Delta M_k = \sqrt{\langle M_k^2 \rangle - \langle M_k \rangle^2} = \sqrt{\langle M_k \rangle (1 - \langle M_k \rangle)}}}$$

ii) Bosonen

$$\langle M_k^2 \rangle = \sum_{M_k=0}^{\infty} M_k^2 p(M_k) = (1-e^{-\mu}) \sum_{M_k=0}^{\infty} M_k^2 e^{-\mu M_k} \quad (3)$$

Vierhan:

$$S = \sum_{M_k=0}^{\infty} e^{-\mu M_k} = \frac{1}{1 - e^{-\mu}}$$

$$\Rightarrow \sum_{M_k=0}^{\infty} M_k^2 e^{-\mu M_k} = \frac{\partial^2 S}{\partial \mu^2} = \frac{\partial}{\partial \mu} \left(\frac{e^{-\mu}}{(1 - e^{-\mu})^2} \right)$$

$$= \frac{e^{\mu}}{(1-e^{\mu})^2} + 2 \frac{(e^{\mu})^2}{(1-e^{\mu})^3}$$

Insert in (3), get the:

$$\langle n_k^2 \rangle = \frac{e^{\mu}}{1-e^{\mu}} + 2 \left(\frac{e^{\mu}}{1-e^{\mu}} \right)^2$$

Now in part a)

$$\langle n_k \rangle = \frac{e^{\mu}}{1-e^{\mu}}$$

Also

$$\langle n_k^2 \rangle = \langle n_k \rangle + 2 \langle n_k \rangle^2$$

So get

$$\underline{\underline{\Delta n_k}} = \sqrt{\langle n_k^2 \rangle - \langle n_k \rangle^2} = \underline{\underline{\sqrt{\langle n_k \rangle (1 + \langle n_k \rangle)}}}$$

d)

i) Fermion

$$\frac{\Delta n_k}{\langle n_k \rangle} = \sqrt{\frac{1 - \langle n_k \rangle}{\langle n_k \rangle}}$$

Maximalavvik för $\langle n_k \rangle$ för fermioner är $\langle n_k \rangle = 1$. Alltså

$$\frac{\Delta n_k}{\langle n_k \rangle} \rightarrow 0 \text{ när } \langle n_k \rangle \rightarrow 1.$$

ii) Bosoner

$$\frac{\Delta n_k}{\langle n_k \rangle} = \sqrt{\frac{1 + \langle n_k \rangle}{\langle n_k \rangle}}$$

Maximalavvik för $\langle n_k \rangle = \infty$. Alltså

$$\frac{\Delta n_k}{\langle n_k \rangle} \rightarrow 1 \text{ när } n_k \rightarrow \infty.$$

ex)

Den rätta sannsynligheten är

$$P(n_1, n_2) = P(0, N) = P(n_1=0) \cdot P(n_2=N)$$

Fra påst. c) har vi

$$P(n_k) = (1 - e^{-\mu - \epsilon_k})^{P(\mu - \epsilon_k)} \cdot e^{-\mu - \epsilon_k}$$

Dette gæi

$$P(M_1=0) = (1 - e^{-\beta\mu})$$

$$P(M_2=N) = (1 - e^{-\beta(\mu-\varepsilon)}) \cdot e^{-\beta(\mu-\varepsilon)N}$$

Altså:

$$\underline{\underline{P(M_1=0; M_2=N) = (1 - e^{-\beta\mu})(1 - e^{-\beta(\mu-\varepsilon)}) e^{-\beta(\mu-\varepsilon)N}}}$$

f)

Fra pkt a) har vi:

$$\langle M_1 \rangle = \frac{1}{e^{-\beta\mu} - 1}$$

$$\langle M_2 \rangle = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}$$

Videre gælder

$$\langle M_1 \rangle + \langle M_2 \rangle = \langle N \rangle$$

og

$$\frac{\langle M_1 \rangle}{\langle M_2 \rangle} = 2$$

Dei to siste likningane gæi

$$\langle M_1 \rangle = \frac{2}{3} \langle N \rangle \quad \text{og} \quad \langle M_2 \rangle = \frac{1}{3} \langle N \rangle$$

Siden $\langle N \rangle \gg 1$, må $\beta\mu$ og $\beta(\mu-\varepsilon)$ nærme seg

Herliker: så gælder

11.

$$\langle M_1 \rangle \approx \frac{-1}{\beta \mu} = \frac{2}{3} \langle N \rangle$$

$$\Rightarrow \mu = -\frac{3}{2\beta \langle N \rangle}$$

$$\langle M_2 \rangle \approx \frac{1}{\beta(\varepsilon - \mu)} = \frac{1}{3} \langle N \rangle$$

Set ind for μ i denne ligning og får:

$$\frac{1}{\beta(\varepsilon + \frac{3}{2\beta \langle N \rangle})} = \frac{1}{3} \langle N \rangle$$

eller

$$\frac{1}{3} \beta \varepsilon \langle N \rangle + \frac{1}{2} = 1$$

$$\Rightarrow \beta = \frac{1}{kT} = \frac{3}{2\varepsilon \langle N \rangle}$$

$$\Rightarrow \underline{\underline{T = \frac{2\varepsilon \langle N \rangle}{3k}}}$$