

TFY Statistisk fysikk
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①

LF

1a

$$M \equiv \sum_i \langle \sigma_i \rangle$$

$$H = -J \sum_i \sigma_i \sigma_{i+1} - h \sum_i \sigma_i$$

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H}$$

$$= \sum_{\{\sigma_i\}} e^{\beta \left(J \sum_i \sigma_i \sigma_{i+1} + h \sum_i \sigma_i \right)}$$

$$M = \frac{1}{Z} \sum_{\{\sigma_i\}} \left(\sum_i \sigma_i \right) e^{\beta \left(J \sum_i \sigma_i \sigma_{i+1} + h \sum_i \sigma_i \right)}$$

$$= \frac{1}{Z} \sum_{\{\sigma_i\}} \frac{\partial}{\partial (\beta h)} e^{\beta \sum_i \left(J \sigma_i \sigma_{i+1} + h \sigma_i \right)}$$

$$= \frac{1}{Z} \frac{\partial}{\partial (\beta h)} Z = \underline{\underline{\left(\frac{\partial \ln Z}{\partial (\beta h)} \right)_T}}$$

b) $Z = \lambda_+^N + \lambda_-^N$ ②

$$\lambda_{\pm} = e^K (\cosh w \pm \sqrt{})$$

$$\begin{aligned}
 M &= \left(\frac{\partial \ln Z}{\partial \beta h} \right)_T \\
 &= \frac{\partial}{\partial \beta h} \left(\ln (\lambda_+^N + \lambda_-^N) \right) \\
 &= \frac{N}{\lambda_+^N + \lambda_-^N} \left(\lambda_+^{N-1} \frac{\partial \lambda_+}{\partial \beta h} + \lambda_-^{N-1} \frac{\partial \lambda_-}{\partial \beta h} \right) \\
 &= \frac{N}{\lambda_+^N + \lambda_-^N} \left\{ \lambda_+^{N-1} e^K \left(\sinh w + \frac{\sinh w \cosh w}{\sqrt{}} \right) \right. \\
 &\quad \left. + N \lambda_-^{N-1} e^K \left(\sinh w - \frac{\sinh w \cosh w}{\sqrt{}} \right) \right\} \\
 &= \frac{N \sinh w}{\lambda_+^N + \lambda_-^N} \frac{1}{\sqrt{}} (\lambda_+^N - \lambda_-^N) \\
 &= \frac{N \sinh w}{\sqrt{}} \left(\frac{\lambda_+^N - \lambda_-^N}{\lambda_+^N + \lambda_-^N} \right)
 \end{aligned}$$

$N \rightarrow \infty$:

$$M \equiv \frac{M}{N} = \frac{\sinh w}{\sqrt{}}$$

c)

③

$$\tau_i \equiv \sigma_i \sigma_{i+1}$$

$$\sigma_i \sigma_{i+2} = \tau_i \tau_{i+1}$$

$$H = - \sum_i (J_1 \tau_i + J_2 \tau_i \tau_{i+1})$$

$J_1 \leftrightarrow h$, $J_2 \leftrightarrow J$ from 1a, 1b

$$\langle \sigma_i \sigma_{i+1} \rangle = \langle \tau_i \rangle \leftrightarrow \langle \sigma_i \rangle \text{ from 1a}$$

$$\langle \sigma_i \sigma_{i+1} \rangle = \frac{\sinh \omega}{\sqrt{\sinh^2 \omega + e^{-4K}}}$$

$$\omega \equiv \beta J_1$$

$$K \equiv \beta J_2$$

$$\langle \sigma_i \sigma_{i+2} \rangle = \langle \tau_i \tau_{i+1} \rangle$$

$$= \frac{1}{N} \sum_i \langle \tau_i \tau_{i+1} \rangle$$

$$= \frac{1}{N} \frac{1}{Z} \frac{\partial}{\partial (\beta J_2)} Z$$

$$= \frac{1}{N} \frac{\partial \ln Z}{\partial (\beta J_2)}$$

Use result for $Z \approx \lambda_+^N$

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$$\lambda_+ = e^K (\cosh \omega + \Gamma)$$

$$\omega = \beta J_1$$

$$K = \beta J_2$$

$$(\sigma_i \sigma_{i+2}) = \frac{1}{N} \frac{\partial \ln Z}{\partial K}$$

$$= \frac{\partial \ln \lambda_+}{\partial K}$$

$$= \frac{1}{\lambda_+} \frac{\partial \lambda_+}{\partial K}$$

$$= \frac{1}{\lambda_+} \left(\lambda_+ - \frac{2 e^{-3K}}{\Gamma} \right)$$

$$= \underline{\underline{1 - \frac{2 e^{-3K}}{\lambda_+ \Gamma}}}$$

(5)

$$i) \quad T \rightarrow 0 ; \quad \beta \rightarrow \infty$$

$$\omega \rightarrow \infty$$

$$K \rightarrow \infty$$

$$\langle \sigma_i \sigma_{i+1} \rangle \rightarrow 1$$

$$\underline{\underline{\langle \sigma_i \sigma_{i+2} \rangle \rightarrow 1}}$$

Spins are aligned at $T \rightarrow 0$.

$$ii) \quad T \rightarrow \infty ; \quad \beta \rightarrow 0$$

$$\omega \rightarrow 0$$

$$K \rightarrow 0$$

$$\lambda_+ \rightarrow 2$$

$$\langle \sigma_i \sigma_{i+1} \rangle \rightarrow 0$$

$$\underline{\underline{\langle \sigma_i \sigma_{i+2} \rangle \rightarrow 0}}$$

At high T , spins are oriented randomly with respect to each other $\Rightarrow$ no correlations.

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(6)

$$Z = \frac{1}{N! h^{dN}} \int d\vec{p}_1 \dots d\vec{p}_N \int d\vec{r}_1 \dots d\vec{r}_N e^{-\beta H}$$

$$H = \sum_{i=1}^N \frac{\vec{p}_i^2}{2m} + \alpha |\vec{r}_i|^d$$

a) Momentum integrals and spatial integrals factorize.

$$Z = \frac{1}{N! h^{dN}} (Z_1)^N$$

$$Z_1 = \underbrace{\int d\vec{p}_1 e^{-\beta \frac{\vec{p}_1^2}{2m}}}_{= (2\pi m k_B T)^{\frac{d}{2}}} \cdot \int d\vec{r}_1 e^{-\beta \alpha |\vec{r}_1|^d}$$

$$= \int d\vec{r}_1 e^{-\beta \alpha |\vec{r}_1|^d}$$

$$= \Omega_d \int_0^R dr r^{d-1} e^{-\beta \alpha r^d}$$

⑦

$$= \frac{\Omega d}{d} \int_0^{R^d} du e^{-\beta \alpha u}$$

$$= \frac{\Omega d}{\beta \alpha d} \int_0^{\beta \alpha R^d} dx e^{-x}$$

$$= \frac{\Omega d}{\beta \alpha d} \left(1 - e^{-\beta \alpha R^d} \right)$$

$$= \underbrace{\frac{\Omega d R^d}{d}}_{\equiv V} \underbrace{\frac{1}{\beta \alpha R^d}}_{=x} \left(1 - e^{-\beta \alpha R^d} \right)$$

$$= V \left(\frac{1 - e^{-x}}{x} \right)$$

$$Z = \frac{1}{N! h^{dN}} \frac{V^N}{\left(\frac{1}{2\pi m h \omega} \right)^{dN}} \left(\frac{1 - e^{-x}}{x} \right)^N$$

$$= \frac{V^N}{N! \lambda^{dN}} \left(\frac{1 - e^{-x}}{x} \right)^N$$

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$$b) \quad U = - \frac{\partial}{\partial \beta} \ln Z$$

$$= \frac{dN}{2} \cdot k_B T - N \frac{\partial}{\partial \beta} \ln \left(\frac{1 - e^{-x}}{x} \right)$$

$$= \frac{dN}{2} \cdot k_B T - N \underbrace{\frac{\partial}{\partial x} \ln \left(\frac{1 - e^{-x}}{x} \right)}_{= \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)} \underbrace{\frac{\partial x}{\partial \beta}}_{= \frac{d\alpha V}{d\beta}}$$

$$= \frac{dN}{2} \cdot k_B T + N \frac{d\alpha V}{d\beta} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$$

$$S = \frac{1}{T} (U - F)$$

U given above

$$F = -k_B T \ln Z$$

$$= -k_B T \left(-\ln(N!) + N \ln V - Nd \ln A + N \ln \left(\frac{1 - e^{-x}}{x} \right) \right)$$

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3)

$$\langle N \rangle = \frac{\partial}{\partial (\beta \mu)} \ln Z_g$$

$$= \sum_{k, \sigma} \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

$$U = \sum_{k, \sigma} \frac{\epsilon_k}{e^{\beta(\epsilon_k - \mu)} + 1}$$

$$\langle N \rangle = 2 \int_0^\infty d\epsilon g(\epsilon) \frac{1}{e^{\beta\epsilon} \frac{1}{z} + 1}$$

$$= 2 \int_0^\infty d\epsilon g(\epsilon) z \frac{e^{-\beta\epsilon}}{1 + z e^{-\beta\epsilon}}$$

$$= 2 \int_0^\infty d\epsilon g(\epsilon) z e^{-\beta\epsilon} \sum_{l=0}^{\infty} (-1)^l z^l e^{-\beta\epsilon l}$$

$$= 2 \sum_{l=0}^{\infty} (-1)^l z^{l+1} \int_0^\infty d\epsilon e^{-\beta\epsilon(l+1)} \cdot \frac{\sqrt{2\pi\epsilon}}{(2\pi\hbar c)^2}$$

$$= \frac{4\pi V}{(2\pi\hbar c)^2} \sum_{l=0}^{\infty} (-1)^l z^{l+1} \underbrace{\frac{1}{(\beta(l+1))^2} \int_0^{\beta} dx x e^{-x}}_{=1}$$

$$= \frac{4\pi V}{(2\pi\hbar c)^2} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{\beta^2 l^2} z^l$$

$$= \sum_{l=1}^{\infty} l \beta_l z^l; \quad \beta_l = \frac{4\pi V}{(2\pi\hbar c)^2} \frac{(-1)^{l+1}}{\beta^2 l^3}$$

$$U = 2 \int_0^{\beta} d\varepsilon g(\varepsilon) z = \frac{\varepsilon e^{-\beta\varepsilon}}{1 + z e^{-\beta\varepsilon}}$$

$$= \frac{4\pi V}{(2\pi\hbar c)^2} \cdot \sum_{l=0}^{\infty} (-1)^l z^{l+1} \underbrace{\int_0^{\beta} d\varepsilon \varepsilon^2 e^{-\beta\varepsilon(l+1)}}_{=2}$$

$$= \frac{1}{\beta^3(l+1)^3} \int_0^{\beta} dx x^2 e^{-x}$$

$$\frac{\beta U}{2} = \frac{4\pi V}{(2\pi\hbar c)^2} \sum_{l=1}^{\infty} (-1)^{l+1} \frac{z^l}{\beta^2 l^3}$$

$$\frac{\beta U}{2} = V \sum_{l=1}^{\infty} b_l z^l$$

$$b_l = \frac{4\pi}{(2\pi\hbar c)^3} \frac{(-1)^{l+1}}{\beta^2 l^3}$$

$$\beta p V = \ln Z$$

$$\langle N \rangle = z \frac{\partial \ln Z}{\partial z} = V \sum_{l=1}^{\infty} l b_l z^l$$

$$\Rightarrow \beta p V = V \sum_{l=1}^{\infty} b_l z^l = \frac{\beta U}{2}$$

$$\frac{U}{pV} = 2$$

5) Pressure at $T=0$

$$pV = \frac{U}{2}$$

$$= \frac{1}{2} \sum_{k, \sigma} \frac{\epsilon_k}{e^{\beta(\epsilon_k - \mu)} + 1}$$

$$= \frac{1}{2} \cdot 2 \cdot \int_0^{\mu} d\epsilon g(\epsilon) \epsilon$$

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$$pV = \frac{V}{(2\pi\hbar c)^2} \cdot 2\pi \int_0^\mu d\varepsilon \varepsilon^2$$

$\frac{\mu^3}{3}$

$$\langle N \rangle = \frac{V}{(2\pi\hbar c)^2} \cdot 2\pi \int_0^\mu d\varepsilon \varepsilon$$

$\frac{\mu^2}{2}$

$$\frac{\langle N \rangle}{V} = \frac{(2\pi\hbar c)^2}{2\pi} \cdot \frac{2}{2} = \mu^2$$

$$\mu = \frac{1}{\sqrt{\pi}} \cdot 2\pi\hbar c \int$$

$$pV = \frac{V}{(2\pi\hbar c)^2} \cdot \frac{2\pi}{\pi^{3/2}} (2\pi\hbar c)^3 \int^3$$

$$p = \frac{2\pi}{\pi^{3/2}} \cdot 2\pi\hbar c \int^3$$

$$\hbar \rightarrow 0 \Rightarrow p = 0$$

c)

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$$\langle N_{\sigma=1} \rangle = \frac{\langle N \rangle}{2} = \langle N_{\sigma=-1} \rangle$$

$$m = \frac{1}{N} (\langle N_{\sigma=1} \rangle - \langle N_{\sigma=-1} \rangle) = 0$$