

TFY4230 Statistisk fysikk.

Kont. eksamen 12.08.2017

Løsningsforslag

①

1a)

$$\begin{aligned}
 \beta p V &= \ln Z_g \\
 &= - \sum_k \ln(1 - e^{-\beta(\epsilon_k - \mu)}) \\
 &= \sum_k f(\epsilon_k)
 \end{aligned}$$

$$f(\epsilon_k) = - \ln(1 - e^{-\beta(\epsilon_k - \mu)})$$

$$\sum_k f(\epsilon_k) = \sum_k \int_{-\infty}^{\infty} d\epsilon \delta(\epsilon - \epsilon_k) f(\epsilon)$$

$$= \int_{-\infty}^{\infty} d\epsilon g(\epsilon) f(\epsilon)$$

$$g(\epsilon) \equiv \sum_k \delta(\epsilon - \epsilon_k)$$

$$\Rightarrow \beta p V = - \int_{-\infty}^{\infty} d\epsilon g(\epsilon) \ln(1 - e^{-\beta(\epsilon - \mu)})$$

$$\langle N \rangle = \frac{\partial (\ln Z_g)}{\partial \beta \mu}$$

$$= - \frac{\partial}{\partial \beta \mu} \sum_k \ln(1 - e^{-\beta(\epsilon_k - \mu)})$$

$$= \sum_k \frac{e^{-\beta(\epsilon_k - \mu)}}{1 - e^{-\beta(\epsilon_k - \mu)}}$$

(2)

$$= \sum_k \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$$

Here, we have to be careful when we convert the sum to an integral, in that we have to single out the term containing the lowest value of ϵ_k , bearing in mind that $\mu < (\epsilon_k)_{\min}$.

Hence,

$$\langle N \rangle = N_0 + \int_{-\infty}^{\infty} d\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon - \mu)} - 1}$$

where $N_0 \equiv \frac{1}{e^{\beta(\epsilon_0 - \mu)} - 1}$

$$\epsilon_0 \equiv (\epsilon_k)_{\min}$$

$$\langle N \rangle = \sum_k \langle n_k \rangle \Rightarrow \langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$$

(3)

$$H = \sum_k \epsilon_k n_k$$

$$U = \langle H \rangle = \sum_k \epsilon_k \langle n_k \rangle$$

$$= \sum_k \frac{\epsilon_k}{e^{\beta(\epsilon_k - \mu)} - 1}$$

$$= \int_{-\infty}^{\infty} d\epsilon \frac{g(\epsilon) \epsilon}{e^{\beta(\epsilon - \mu)} - 1}$$

Note that in this case, we have no additional separate term involving ϵ_0 , since $\epsilon_0 = 0$.

$$\underline{\underline{5)}} \quad \beta pV = - \int_{-\infty}^{\infty} d\epsilon g(\epsilon) \ln(1 - e^{-\beta(\epsilon - \mu)})$$

We transform this expression by a partial integration, to obtain

$$\beta pV = \int_{-\infty}^{\infty} d\epsilon \frac{G(\epsilon)}{e^{\beta(\epsilon - \mu)} - 1} \quad \text{NB!!}$$

$$G'(\epsilon) = g(\epsilon)$$

(4)

Here, $g(\epsilon) = A \epsilon \Theta(\epsilon)$

$$Q(\epsilon) = \frac{1}{2} g(\epsilon) \epsilon$$

Hence

$$pV = \frac{1}{2} \int_{-\infty}^{\infty} d\epsilon \frac{g(\epsilon) \epsilon}{e^{\beta(\epsilon-\mu)} - 1} = \frac{U}{2}$$

$$\underline{\underline{K=2}}$$

c)

$$N_0 = \langle N \rangle - \int_{-\infty}^{\infty} d\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} - 1}$$

At $T=0$, $N_0 = \langle N \rangle$, all particles are in the lowest-energy state.

As T increases, N_0 is reduced, eventually reaching the value $N_0=0$ as $T \rightarrow T_c$. Thus, at $T=T_c$, we have:

$$\begin{aligned} \langle N \rangle &= \int_{-\infty}^{\infty} d\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} - 1} \\ &= V \frac{2\pi}{(hc)^2} \int_0^{\infty} d\epsilon \frac{\epsilon}{e^{\beta\epsilon} - 1} \end{aligned}$$

$$\int_0^{\infty} d\varepsilon \frac{\varepsilon}{e^{\beta\varepsilon} - 1} = \frac{1}{\beta^2} \int_0^{\infty} dx \frac{x}{e^x - 1}$$

$$= \frac{1}{\beta^2} \zeta(2) \Gamma(2)$$

$$\mathcal{G} = \frac{2\pi}{(\beta hc)^2} \zeta(2) \Gamma(2)$$

We may now define a length scale in the problem, an ultrarelativistic version of the thermal wavelength

$$\lambda_T \equiv \beta c$$

Then the BEC criterion is given by

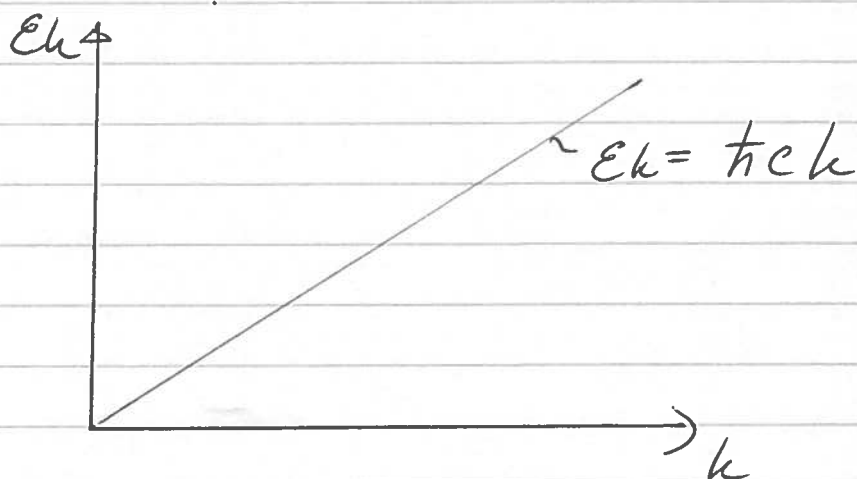
$$\lambda_T^2 \mathcal{G} = 2\pi \zeta(2) \Gamma(2) \sim \mathcal{O}(1)$$

$$(k_B T \lambda)^2 = \frac{1}{2\pi \zeta(2) \Gamma(2)} \mathcal{G} (hc)^2$$

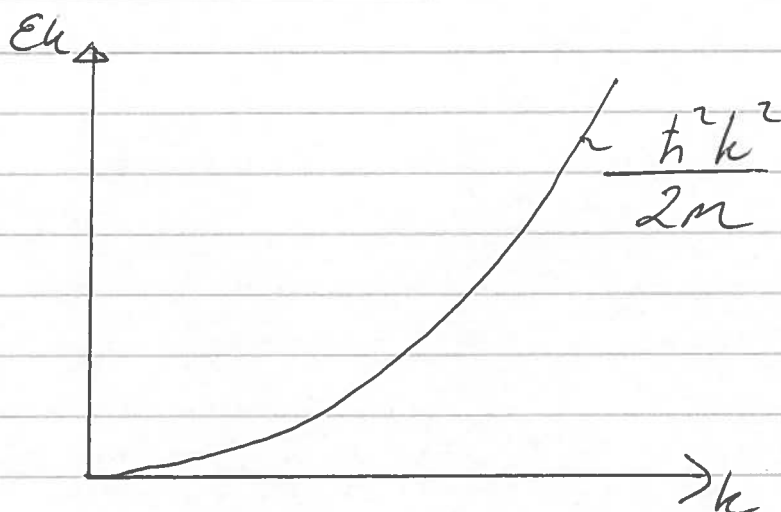
$$k_B T \lambda = \frac{\mathcal{G}^{1/2}}{\sqrt{2\pi \zeta(2) \Gamma(2)}} hc$$

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d) ultra-rel. case:



Non-rel. case:



The energy required to excite particles out of the ground state is E_k . In 2D, the DOS $g(E) \sim E$ for ultra-rel. particles, and $g(E) \sim \text{constant}$ for non-rel. particles. Thus, there are infinitely more states available at low energies in the non-rel. case compared to the ultra-rel. case. This is the origin of the difference in the BEC-behavior at low T the two cases.

(7)

2a

$$Z = Z_1^N \quad \text{Independent spins}$$

$$Z_1 = \sum_{\sigma_i = \pm 1, 0} e^{\beta h S \sigma_i}$$

$$= 1 + e^{\beta h S} + e^{-\beta h S}$$

$$= 1 + 2 \cosh(\beta h S)$$

$$\underline{Z = \left(1 + 2 \cosh(\beta h S)\right)^N}$$

$$b) \quad H_c = - \frac{\partial \ln Z}{\partial \beta} = -N \frac{2hS \sinh(\beta h S)}{1 + 2 \cosh(\beta h S)}$$

$$= -2NhS \frac{\sinh x}{1 + 2 \cosh x}; \quad x \equiv \beta h S$$

2c

$$C_h = \left(\frac{\partial H_e}{\partial T} \right)_h = - \frac{1}{k_B T^2} \frac{\partial H_e}{\partial \beta}$$

$$H_e = - \frac{\partial \ln Z}{\partial \beta} = k_B T^2 \frac{\partial \ln Z}{\partial T}$$

$$C_h = \frac{1}{k_B T^2} \frac{\partial^2 \ln Z}{\partial \beta^2}$$

$$= k_B \beta^2 \frac{\partial^2 \ln Z}{\partial \beta^2}$$

$$= k_B \beta^2 \frac{\partial}{\partial \beta} \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right)$$

$$= k_B \beta^2 \left[\frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right)^2 \right]$$

$$\left. \begin{aligned} \frac{1}{Z} \frac{\partial Z}{\partial \beta} &= - \langle H \rangle \\ \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} &= \langle H^2 \rangle \end{aligned} \right\} \underline{C_h = k_B \beta^2 [\langle H^2 \rangle - \langle H \rangle^2]}$$

Q. E. D.

(9)

2d) The simplest way of doing this is to use the basic expression

$$C_h = \left(\frac{\partial H_e}{\partial T} \right)_h = -k_B \beta^2 \frac{\partial H_e}{\partial \beta}$$

and then use an approximation for H_e valid when $\beta h S \gg 1$

$$H_e = -2 N h S \frac{\sinh x}{1 + 2 \cosh x}$$

$$\approx -N h S \tanh x$$

For large x , $\tanh x$ approaches 1 from below, but since we are going to take a derivative to obtain C_h , we need to now precisely how $\tanh x$ approaches 1.

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} \approx 1 - 2e^{-2x}$$

$$C_h = -k_B \beta^2 h S \frac{\partial H_e}{\partial x}$$

$$= k_B \beta^2 (h S)^2 N 4 e^{-2\beta h S}$$

$$= \underline{\underline{N k_B (2\beta h S)^2 e^{-2\beta h S}}}$$

(10)

The heat capacity vanishes essentially exponentially as T is lowered to very low T ($\beta h S \gg 1$).

Physical explanation: Heat capacity originates with energy-fluctuations (problem 2c).

Energy-fluctuations originate with spin-fluctuations in this model

In this generalized Ising-model, spins can take three discrete values $\sigma_i = (0, \pm 1)S$. The energy difference between each state is hS , and this is roughly the minimum amount of thermal energy $k_B T$ that must be supplied in order to flip a spin. This energy is not available when $k_B T \ll hS$ or equivalently $\beta h S \gg 1$. Thus, when $k_B T \ll hS$, there are no spin-fluctuations $\Rightarrow$ no energy-fluctuations $\rightarrow C_h \rightarrow 0$ (exponentially)

3a

$$U = \langle H \rangle = \frac{1}{N! h^{2N}} \frac{\int \dots \int d\Gamma H e^{-\beta H}}{Z}$$

$$= - \frac{1}{Z} \frac{\partial Z}{\partial \beta} = - \frac{\partial \ln Z}{\partial \beta}$$

b)

$$Z = \frac{1}{N!} \frac{1}{h^{2N}} (Z_p)^N (Z_{it})^N$$

$$Z_p = \int d^2 p e^{-\frac{\beta p^2}{2m}}$$

$$= 2\pi \int_0^\infty dp p e^{-\frac{\beta p^2}{2m}}$$

$$= \pi \int_0^\infty dx e^{-\frac{\beta}{2m} x^2}$$

$$= \frac{2\pi m}{\beta} = \frac{2\pi m k_B T}{\beta}$$

$$Z_{it} = \int_V d^2 r e^{-\beta V_0 r^2} = Q_1$$

$$= 2\pi \int_0^R dr r e^{-\frac{\beta V_0 r^2}{2}} ; \quad \beta V_0 \equiv \frac{1}{R_0^2}$$

Thus, we have:

$$Z = \frac{1}{N!} \frac{1}{h^{2N}} (2\pi m k_B T)^N Q_1^N$$

$$= \frac{1}{\lambda^{2N} N!} Q_1^N ; \lambda \equiv \frac{h}{\sqrt{2\pi m k_B T}}$$

c) Explicit computation of Q_1 :

$$Q_1 = \pi \int_0^{R^2} dx e^{-\frac{x}{R_0^2}}$$

$$= \pi R_0^2 \int_0^{\frac{R^2}{R_0^2}} du e^{-u}$$

$$= \pi R_0^2 \left(1 - e^{-\frac{R^2}{R_0^2}} \right)$$

$$p = k_B T \left(\frac{\partial \ln Z}{\partial V} \right)_T = N k_B T \left(\frac{\partial \ln Q_1}{\partial V} \right)_T$$

$$= \frac{N k_B T}{\pi} \frac{\partial \ln Q_1}{\partial R^2}$$

$$p = \frac{Nk_B T}{\pi R^2} \frac{1}{\int R^2} \ln \left(1 - e^{-\frac{R^2}{R_0^2}} \right)$$

$$= \frac{Nk_B T}{\pi R_0^2} \frac{1}{e^{\frac{R^2}{R_0^2}} - 1}$$

$$\frac{R}{R_0} \ll 1 \Rightarrow e^{\frac{R^2}{R_0^2}} - 1 \approx \frac{R^2}{R_0^2}$$

$$p = \frac{Nk_B T}{\pi R^2} = \frac{Nk_B T}{V} \quad \text{ideal gas law}$$

$$\frac{R}{R_0} \gg 1 \Rightarrow e^{\frac{R^2}{R_0^2}} - 1 \approx e^{\frac{R^2}{R_0^2}}$$

$$p = \frac{Nk_B T}{\pi R_0^2} e^{-\frac{R^2}{R_0^2}}$$

d) The total internal energy is given by

$$U = - \frac{\partial \ln Z}{\partial \beta}$$

$$= -N \frac{\partial \ln Z_p}{\partial \beta} - N \frac{\partial \ln Q_i}{\partial \beta}$$

$$= N k_B T + N k_B T - N \frac{\partial}{\partial \beta} \ln \left(1 - e^{-\frac{R^2}{R_0^2}} \right)$$

$$= 2 N k_B T - N \frac{V_0 R^2}{e^{\frac{R^2}{R_0^2}} - 1}$$

$$= \langle H \rangle$$

$$= N \left\langle \frac{p_{ci}^2}{2m} + \frac{p_{ci}^2}{2m} + V_0 r_{ci}^2 \right\rangle$$

$$= N \left[\frac{k_B T}{2} + \left\langle \frac{p_{ci}^2}{2m} + V_0 r_{ci}^2 \right\rangle \right]$$

$$\left\langle \frac{p_{ci}^2}{2m} + V_0 r_{ci}^2 \right\rangle = \frac{3 N k_B T}{2} - \frac{N V_0 R^2}{e^{\frac{R^2}{R_0^2}} - 1}$$