

①

Eksamen TFY4230 Stat. fys
02.06.2018

1a) $H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i$

$$\sigma_i = m + \delta\sigma_i \Rightarrow \langle \sigma_i \rangle = m$$

$$\delta\sigma_i = \sigma_i - m$$

$$H = -J \sum_{\langle i,j \rangle} (m + \delta\sigma_i)(m + \delta\sigma_j) - B \sum_i (m + \delta\sigma_i)$$

$$\approx -Jm^2 Nz - Jmz \sum_i \delta\sigma_i - Jmz \sum_j \delta\sigma_j$$

$$- BmN - B \sum_i \delta\sigma_i$$

$$= -Jm^2 Nz + 2Jm^2 Nz - 2Jmz \sum_i \sigma_i - B \sum_i \sigma_i$$

$$= Jm^2 Nz - (B + 2Jmz) \sum_i \sigma_i$$

$$= \underline{Jm^2 Nz - B_{\text{eff}} \sum_i \sigma_i} \quad \text{g. e. d.} \quad (*)$$

b) $Z = \sum_{\{\sigma_i\}} e^{-\beta H} = e^{-\beta G}$

$$= \left[\sum_{\sigma_i} e^{-\beta (Jm^2 z - B_{\text{eff}} \sigma_i)} \right]^N$$

$$= Z_1$$

(*) B_{eff} is an effective magnetic field felt by each spin from external B and surrounding spins, m is the magnetization of the system

(2)

$$Z_1 = \sum_{\sigma=0, \pm 1, \pm 2} e^{-\beta J m^2 z + \beta B \mu \sigma}$$

$$= e^{-\beta J m^2 z} (1 + 2 \cosh \Omega + 2 \cosh 2\Omega)$$

$$\Omega \equiv \beta B \mu = \beta (B + 2Jzm) = \omega + \alpha m$$

$$\omega \equiv \beta B ; \alpha \equiv 2Jz\beta$$

$$\text{Define } Y(\omega + \alpha m) = 2(\cosh \Omega + \cosh 2\Omega)$$

$$Z = e^{-\beta G} = e^{-\beta J m^2 z N} (1 + Y)^N$$

$$= e^{-\beta J m^2 z N + N \ln(1+Y)} \Rightarrow$$

$$G = J m^2 z N - \frac{N}{\beta} \ln(1+Y) ; \text{g.e.d.}$$

$$Y = Y(\omega + \alpha m) = Y(\beta B + 2\beta J z m)$$

$$\text{Consider } H: B \rightarrow -B ; \sigma_i \rightarrow -\sigma_i$$

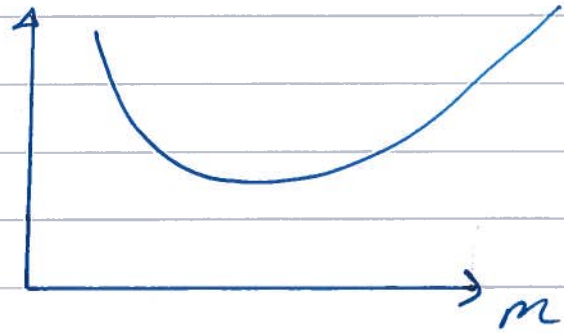
$$\Rightarrow m \rightarrow -m \Rightarrow \omega + \alpha m \rightarrow -(\omega + \alpha m)$$

Under this transformation, H is invariant (does not change). It corresponds to "flipping" the system upside-down. This does not change the physics, when the field B is also "flipped" upside-down. Thus, G , and hence Y is invariant when $x \rightarrow -x$.

(3)

c) m is a so far unknown parameter that must be determined self-consistently by choosing the m that minimizes G .

$$\textcircled{a} \quad \boxed{\frac{\partial G}{\partial m} = 0}$$



$$\frac{\partial G}{\partial m} = \frac{\partial}{\partial m} \left[J m^2 N z - \frac{N}{\beta} \ln [1 + Y(\omega + \alpha m)] \right]$$

$$= 2 J m z N - \frac{N}{\beta} \frac{1}{1+Y} \cdot \frac{\partial}{\partial m} Y(\omega + \alpha m)$$

$$= 2 J N z m - \frac{N}{\beta} \alpha \frac{Y'(\omega + \alpha m)}{1 + Y(\omega + \alpha m)}$$

Inserting the expression for α ,
dividing by $2 J N z$, and using (*),
we find

$$m = \frac{Y'(\alpha + \omega m)}{1 + Y(\alpha + \omega m)}$$

(4)

d) T_c is defined only when $B=0$!
 If $B \neq 0$, there is no special T
 that separates the low-temp.
 phase from the high-temp. phase.
 So we set $B=0$. Then we get

$$m = \frac{Y'(\alpha m)}{1 + Y(\alpha m)} ; \quad \omega = 0$$

$$Y = 2 (\cosh(\alpha m) + \cosh(2\alpha m))$$

$$Y' = 2 [\sinh(\alpha m) + 2\sinh(2\alpha m)]$$

$$T \rightarrow T_c^- : m \ll 1 \Rightarrow$$

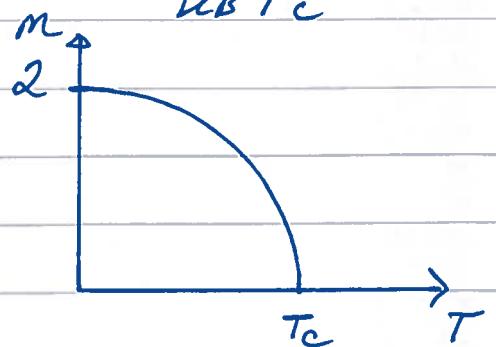
$$Y'(\alpha m) \approx 2 (\alpha m + 2\alpha m \cdot 2)$$

$$= 2.5 \alpha m \quad (\Rightarrow Y''(0) = 10)$$

$$m \approx \frac{10 \alpha m}{1 + 4} = 2 \alpha m$$

$$1 = 2 \cdot 2 Jz \beta_c ; \quad \beta_c = \frac{1}{k_B T_c}$$

$$\underline{k_B T_c = 4 Jz}$$

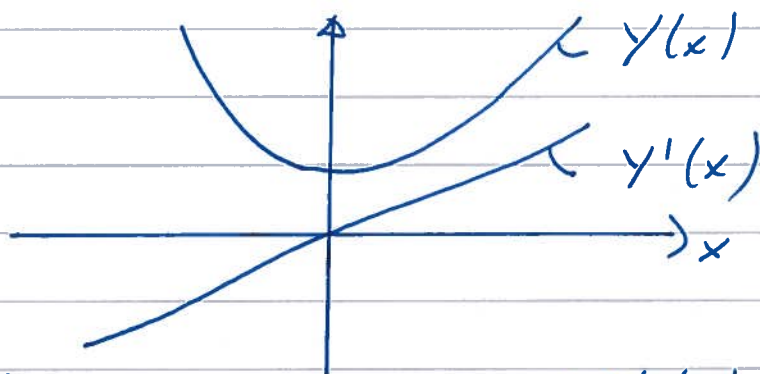


(5)

Even if we had not had an explicit expression for Y , we could still make progress.

$$Y(0) = 4$$

$Y(x) = Y(-x) \Rightarrow Y(x)$ is an even function of x , $\Rightarrow Y'(x)$ is an odd function of x .



$Y'(x=0) = 0$ and $Y'(x)$ must be linear in x when $|x| \ll 1$

$$Y'(x) \cong Y''(0) x ; x \ll 1$$

$$m \cong \frac{Y''(0)}{5} \alpha m$$

$$1 = \frac{Y''(0)}{5} 2Jz \beta_c$$

$$\underline{k_B T_c = \frac{2 Y''(0)}{5} Jz \quad (= 4Jz)}$$

⑥

2a) $Z_g = e^{\beta p V} \Rightarrow$

$$\beta p V = \sum_k \ln (1 + e^{-\beta(\epsilon_k - \mu)})$$

$$= \int d\epsilon g(\epsilon) \ln (1 + e^{-\beta(\epsilon - \mu)})$$

$$g(\epsilon) = \sum_k \delta(\epsilon - \epsilon_k)$$

$$\beta p V = V \frac{\Omega d}{(hc)^d} \int d\epsilon \epsilon^{d-1} \ln (1 + e^{-\beta(\epsilon - \mu)})$$

$$= V \frac{\Omega d}{(hc)^d} \left\{ \int_0^\infty \frac{1}{d} \epsilon^d \ln (1 + e^{-\beta(\epsilon - \mu)}) \right.$$

$$\left. - \frac{1}{d} \int_0^\infty d\epsilon \epsilon^d \frac{(-\beta) \cdot e^{-\beta(\epsilon - \mu)}}{1 + e^{-\beta(\epsilon - \mu)}} \right\}$$

$$\beta p V = V \beta \frac{\Omega d}{(hc)^d} \frac{1}{d} \int_0^\infty d\epsilon \frac{\epsilon^d}{e^{\beta(\epsilon - \mu)} + 1}$$

$$p = \frac{\Omega d}{(hc)^d} \frac{1}{d} \int_0^\infty d\epsilon \frac{\epsilon^d}{e^{\beta(\epsilon - \mu)} + 1} \quad \text{q.e.d.}$$

(7)

$$\begin{aligned}
\langle N \rangle &= \frac{\partial}{\partial (\beta \mu)} \ln Z_g \\
&= \frac{\partial}{\partial (\beta \mu)} \int d\varepsilon g(\varepsilon) \ln(1 + e^{-\beta(\varepsilon - \mu)}) \\
&= V \frac{\Omega d}{(hc)^d} \int_0^\infty d\varepsilon \varepsilon^{d-1} \frac{e^{-\beta(\varepsilon - \mu)}}{1 + e^{-\beta(\varepsilon - \mu)}}
\end{aligned}$$

$$g = \frac{\Omega d}{(hc)^d} \int_0^\infty d\varepsilon \frac{\varepsilon^{d-1}}{e^{\beta(\varepsilon - \mu)} + 1} \quad \text{q.e.d.}$$

b)

$$\begin{aligned}
\rho &= \frac{\Omega d}{(hc)^d} \frac{1}{d} \int_0^\infty d\varepsilon \varepsilon^d \sum_{l=1}^\infty (-1)^{l+1} e^{-\beta(\varepsilon - \mu)l} \\
&= \frac{\Omega d}{(hc)^d} \frac{1}{d} \sum_{l=1}^\infty (-1)^{l+1} z^l \underbrace{\int_0^\infty d\varepsilon \varepsilon^d e^{-\beta \varepsilon l}}
\end{aligned}$$

$$= \frac{1}{\beta^{d+1} l^{d+1}} \Gamma(d+1)$$

$$\beta \rho = \frac{\Omega d}{(\beta hc)^d} \underbrace{\frac{\Gamma(d+1)}{d}}_{= \Gamma(d)} \sum_{l=1}^\infty \frac{(-1)^{l+1}}{l^{d+1}} z^l$$

(8)

$$\begin{aligned}
g &= \frac{\Omega d}{(hc)^d} \int_0^\infty d\varepsilon \varepsilon^{d-1} \sum_{l=1}^\infty (-1)^{l+1} e^{-\beta(\varepsilon - \mu)l} \\
&= \frac{\Omega d}{(hc)^d} \sum_{l=1}^\infty (-1)^{l+1} z^l \underbrace{\int_0^\infty d\varepsilon \varepsilon^{d-1} e^{-\beta \varepsilon l}}_{= \frac{1}{\beta^d l^d} \Gamma(d)} \\
&= \frac{\Omega d}{(\beta hc)^d} \Gamma(d) \sum_{l=1}^\infty \frac{(-1)^{l+1}}{l^d} z^l
\end{aligned}$$

Comparing with the given expressions, we find

$$b_l = \frac{(-1)^{l+1}}{l^{d+1}}$$

$$\frac{1}{z^d} = \frac{\Omega d \Gamma(d)}{(\beta hc)^d}$$

$$z = \beta hc \left(\frac{1}{\Omega d \Gamma(d)} \right)^{1/d}$$

(9)

Alternatively, we could have defined

$$b_l = \frac{(-1)^{l+1}}{l^{d+1}} \Omega_d \Gamma(d)$$

$$\underline{\underline{\lambda = \beta \hbar c}}$$

c) $\beta p = \frac{1}{\lambda^d} F(z)$

$$g = \frac{1}{\lambda^d} G(z) = \frac{1}{\lambda^d} F'(z)$$

$$z = G^{-1}(g \lambda^d)$$

$$\beta p = \frac{1}{\lambda^d} F(G^{-1}(g \lambda^d))$$

$$= \frac{1}{\lambda^d} H(g \lambda^d)$$

$$\beta p = \frac{1}{\lambda^d} \left(z - \frac{z^2}{2^{1+d}} + \dots \right)$$

$$g = \frac{1}{\lambda^d} \left(z - \frac{z^2}{2^d} + \dots \right)$$

$$\beta p - \rho = \frac{1}{\lambda^d} z^2 \frac{1}{2^d} \left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{2^{d+1}} \frac{z^2}{\lambda^d}$$

$$z \approx \rho \lambda^d$$

$$\beta p - \rho = \frac{1}{2^{d+1}} \rho^2 \lambda^d$$

$$\beta p = \rho + B_2 \rho^2 + \dots$$

$$B_2 = \frac{1}{2^{d+1}} \lambda^d$$

$$\beta p = \rho \left(1 + \frac{1}{2^{d+1}} \lambda \rho^d + \dots\right)$$

Parameter that controls the quantum corrections to ideal gas law: $\frac{\rho \lambda^d}{2^{d+1}}$

(11)

$g \lambda^d$ becomes sizeable when the thermal wavelength becomes comparable to separation, λ , between particles $g \sim \frac{1}{\lambda^d}$

$\frac{\lambda}{r} \sim 1 \Rightarrow$ corrections become large.

Quantum corrections become important when wavefunctions of particles start to overlap.

Moreover, the positive correction to the pressure from the $B_2(T) g^2$ -term means that, somehow, the overlap of wavefunctions of different particles represent a repulsion between them.

Such "repulsion" between single-particle quantum states is characteristic of fermionic systems. This is what we would expect from the expression for Z_g :

$$Z_g = \prod_k \left(1 \pm e^{-\beta(E_k - \mu)} \right)^{\pm 1} \quad \begin{array}{l} +1: \text{Fermions} \\ -1: \text{Bosons} \end{array}$$

$$3a) \quad Z = \frac{1}{h^N} \frac{1}{N!} (Z_1)^N$$

$$Z_1 \equiv \int_{-\infty}^{\infty} dp e^{-\frac{\beta p^2}{2m}} \int_{-\infty}^{\infty} dx e^{-\beta \left(\frac{m\omega^2}{2} x^2 + \alpha x^4 \right)}$$

$$\approx \sqrt{\frac{2\pi m}{\beta}} \int_{-\infty}^{\infty} dx e^{-\frac{\beta p^2}{2m}} (1 - \beta \alpha x^4)$$

$$= \sqrt{\frac{2\pi m}{\beta}} \left(\sqrt{\frac{2\pi}{\beta m \omega^2}} - \beta \alpha \frac{3}{4} \sqrt{\pi} \frac{1}{\gamma^{5/2}} \right)$$

$$= \left(\frac{2\pi}{\beta \omega} \right) \left(1 - \frac{3}{4} \frac{\beta \alpha}{\gamma^2} \right) \Rightarrow$$

$$\underline{Z = \frac{1}{h^N} \frac{1}{N!} \left(\frac{2\pi}{\beta \omega} \right)^N \left(1 - \frac{3}{4} \frac{\beta \alpha}{\gamma^2} \right)^N}$$

This approximation is valid to linear order in α ,
i.e. when $1 \gg \frac{3}{4} \frac{\beta \alpha}{\gamma^2} \Leftrightarrow k_B T \ll \frac{m^2 \omega^2}{\alpha}$

$$b) \quad u = \langle H \rangle = \frac{1}{Z} \frac{1}{h^N} \frac{1}{N!} \int d\Gamma H e^{-\beta H}$$

$$= - \frac{1}{Z} \frac{\partial Z}{\partial \beta} = - \frac{\partial \ln Z}{\partial \beta}$$

(13)

$$U = -N \frac{\partial \ln(1/\beta)}{\partial \beta} - N \frac{\partial}{\partial \beta} \ln \left(1 - \frac{3}{4} \frac{\beta \alpha}{\gamma^2} \right)$$

$$= N k_B T - N \frac{1}{\left(1 - \frac{3}{4} \frac{\beta \alpha}{\gamma^2} \right)} \cdot \left(-\frac{3}{4} \right) \frac{\alpha}{\left(\frac{m\omega^2}{2} \right)^2} \left(-\frac{1}{\beta^2} \right)$$

$$\approx N k_B T \left(1 - \frac{3}{4} \frac{\beta \alpha}{\gamma^2} \right); \quad \underline{\underline{\frac{3}{4} \frac{\beta \alpha}{\gamma^2} \ll 1}}$$

The result in 3a is reasonable when

$$\frac{3}{4} \frac{\beta \alpha}{\gamma^2} \ll 1$$

and the expansion of the exponential is only valid to linear order in α .

$$U = N k_B T - N k_B T \frac{3}{4} \frac{\alpha}{\left(\frac{m\omega^2}{2} \right)^2} k_B T$$

$$\underline{\underline{C_V = N k_B \left(1 - \frac{3}{2} \frac{\beta \alpha}{\gamma^2} \right); \quad \frac{3}{2} \frac{\beta \alpha}{\gamma^2} \ll 1}}$$

c) $U = \langle H \rangle = \langle E_{kin} + U_p \rangle$

$\langle E_{kin} \rangle = \frac{Nk_B T}{2}$ by the classical equipartition theorem

$$\begin{aligned} \langle U_p \rangle &= U - \frac{Nk_B T}{2} \\ &= Nk_B T \left(\frac{1}{2} - \frac{3}{4} \frac{\beta \alpha}{\gamma^2} \right) \end{aligned}$$

Note that we cannot apply the generalized equipartition theorem to $\langle U_p \rangle$ itself, since the coordinate x_i appears in two places in the Hamiltonian, namely $\frac{1}{2} m \omega^2 x_i^2$ and αx_i^4 .