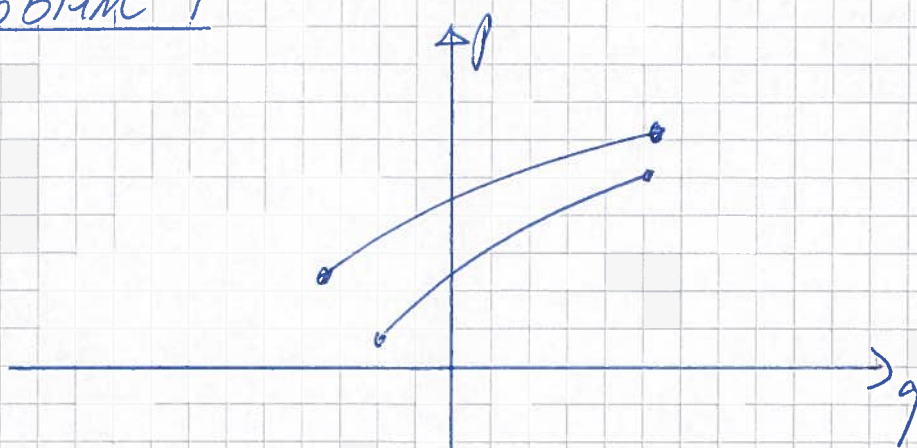


Problem 1

a)



$$\dot{p} = - \frac{\partial H}{\partial q}$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\left. \begin{array}{l} \dot{p} = - \frac{\partial H}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial p} \end{array} \right\} \Rightarrow \frac{dp}{dq} = - \frac{\partial H / \partial q}{\partial H / \partial p}$$

Slope of a given line in phase-space.

If two lines approach each other, then their slopes will also approach each other. Hence, the lines will now cross.

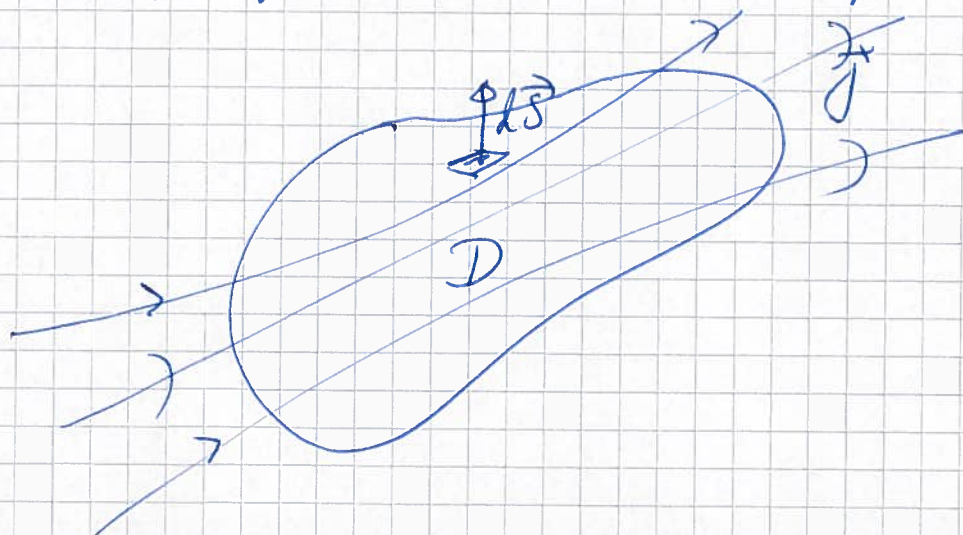
b) This means that points in phase-space now vanish and now appear.

Regarding each point in analogy with a particle, we have a continuity equation for these points in phase-space:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

where ρ is density of points in phase-space

This comes about as follows: Since no points in (p, q) disappear or appear, it means that any change of the # of points in a sub-domain D of (p, q) must be due to a net flow of points in or out of D .



$$\frac{d}{dt} \iiint_D dV \rho(p, q, t) = - \oint_S d\vec{S} \cdot \vec{J}$$

S is a closed surface encompassing D

$$\iiint_D dV \frac{\partial \rho}{\partial t} = - \iiint_D dV (\vec{\nabla} \cdot \vec{J}) \quad \text{Gauss' theorem}$$

Should be valid for any $D \Rightarrow$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$\vec{J} = \rho \vec{v}$, where $\vec{v}$ is velocity in phase space (p, q) . Q.E.D.

$$c) \quad \frac{\partial \mathcal{L}}{\partial t} + \sum_i \left(\frac{\partial}{\partial q_i} (\mathcal{L} \dot{q}_i) + \frac{\partial}{\partial p_i} (\mathcal{L} \dot{p}_i) \right) = 0$$

$$\vec{v} \cdot (\mathcal{L} \vec{x}) = \vec{v}_q \cdot (\mathcal{L} \dot{\vec{q}}) + \vec{v}_p \cdot (\mathcal{L} \dot{\vec{p}})$$

Consider now the terms inside $\sum_i$:

$$\frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + \mathcal{L} \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial p_i} \dot{p}_i$$

$$= \frac{\partial \mathcal{L}}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \mathcal{L}}{\partial p_i} \frac{\partial H}{\partial q_i}$$

$$+ \mathcal{L} \left(\underbrace{\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i}}_{=0} \right)$$

Define:

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$$

Then we obtain

$$\frac{\partial \mathcal{L}}{\partial t} + \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \mathcal{L}}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$= \underline{\underline{\frac{\partial \mathcal{L}}{\partial t} + \{ \mathcal{L}, H \} = 0}}$$

$$d) \quad \frac{d\rho}{dt} = 0 \Rightarrow$$

$$\{\rho, H\} = 0$$

$$\rho = \rho(\{q_i, p_i\})$$

If we now assume that

$$\rho(\{q_i, p_i\}) = \rho(B(\{q_i, p_i\}))$$

then $\{\rho, H\}$

$$= \frac{d\rho}{dB} \{B, H\}$$

which we require to be zero.

Choosing $B = H$ guarantees this.

$\Rightarrow$ we can choose

$$\underline{\rho(\{q_i, p_i\}) = \rho(H)}$$

ρ can be regarded as a probability distribution of states in phase-space if we normalize it. Then, we can compute physically observable quantities $O(\{q_i, p_i\})$ as averages over ρ :

$$\langle O \rangle = \sum_{\{g, \mu, i\}} g(H) O$$

as soon as we specify the Hamiltonian (a microscopic description) of the system. This provides the connection

Microscopic physics $\Rightarrow$ macroscopic physics
 More generally, given H and $g(H)$

$$\langle O \rangle = \sum_{\text{states}} g(H) O$$

which is also suitable for quantum mechanical systems.

Problem 2

a) $M = \sum_i \langle \sigma_i \rangle = \langle \sum_i \sigma_i \rangle$

$$\langle \sum_i \sigma_i \rangle = \frac{1}{Z} \sum_{\{\sigma_i\}} \left(\sum_i \sigma_i \right) e^{-\beta H}$$
$$= \frac{1}{Z} \sum_{\{\sigma_i\}} \frac{\partial}{\partial \beta B} \left(e^{-\beta H} \right)$$
$$= \frac{1}{Z} \frac{\partial}{\partial \beta B} Z = \underline{\underline{\frac{\partial \ln Z}{\partial \beta B}}}$$

$$H_e = \langle H \rangle$$
$$= \frac{1}{Z} \sum_{\{\sigma_i\}} H e^{-\beta H}$$
$$= \frac{1}{Z} \sum_{\{\sigma_i\}} \left(- \frac{\partial}{\partial \beta} \right) e^{-\beta H}$$
$$= - \frac{1}{Z} \frac{\partial Z}{\partial \beta} = - \underline{\underline{\frac{\partial \ln Z}{\partial \beta}}}$$

b) $Z = (Z_{1\text{-dim}})^{N_d}$

since the dimers are independent,
where $Z_{1\text{-dim}}$ is the partition
function of a single spin-dimer.

$$Z_{1\text{-dim}} = \sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1} e^{\beta(J\sigma_1\sigma_2 + B(\sigma_1 + \sigma_2))}$$

$$= \frac{e^{\beta(J+2B)} + e^{\beta(J-2B)} + 2e^{-\beta J}}{}$$

$$Z = (Z_{1\text{-dim}})^{Nd} \quad ; \text{ independent dims}$$

$$H_c = - \frac{\partial \ln Z}{\partial \beta} = - Nd \frac{\partial \ln Z_{1\text{-dim}}}{\partial \beta}$$

$$= - Nd \frac{A}{K}$$

$$K = \frac{e^{\beta(J+2B)} + e^{\beta(J-2B)} + 2e^{-\beta J}}{}$$

$$A = \frac{(J+2B)e^{\beta(J+2B)} + (J-2B)e^{\beta(J-2B)} - 2Je^{-\beta J}}{}$$

K, A may also be written:

$$K = 2 [e^{\beta J} \cosh(2\beta B) + e^{-\beta J}]$$

$$A = 2 [J \cosh(2\beta B) + 2B \sinh(2\beta B) - J e^{-\beta J}]$$

$$\text{Hence, } H_c = - Nd \frac{(J \cosh(2\beta B) + 2B \sinh(2\beta B) - J e^{-\beta J})}{e^{\beta J} \cosh(2\beta B) + e^{-\beta J}}$$

Specific heat:

$$C_B = \left(\frac{\partial H_c}{\partial T} \right)_B = - k_B \beta^2 \left(\frac{\partial H_c}{\partial \beta} \right)_B$$

$$\text{Define } X' \equiv \left(\frac{\partial X}{\partial \beta} \right)_B$$

$$C_B = -N_d (-k_B) \beta^2 H_c'$$

$$= N_d k_B \beta^2 \left(\frac{A'K - AK'}{K^2} \right)$$

Define $x = e^{\beta(J+2B)}$; $y = e^{\beta(J-2B)}$; $z = e^{-\beta J}$

$$a = J+2B; \quad b = J-2B; \quad c = J$$

$$A = ax + by - 2cz; \quad A' = a^2x + b^2y + 2c^2z$$

$$K = x + y + 2z; \quad K' = A$$

$$A'K - AK' = A'K - A^2$$

$$= (a^2x + b^2y + 2c^2z)(x + y + 2z)$$

$$- (ax + by - 2cz)(ax + by - 2cz)$$

$$= \frac{xy(a-b)^2 + 2xz(a+c)^2 + 2yz(b+c)^2}{2\beta J}$$

$$a-b = 4B$$

$$a+c = 2(J+B)$$

$$b+c = 2(J-B)$$

$$xy = e^{2\beta B}$$

$$xz = e^{-2\beta B}$$

$$yz = e^{2\beta J}$$

$$yz = e^{2\beta J}$$

$$A'K - A^2 = \frac{16B^2 e^{2\beta J} + 8(J+B)^2 e^{2\beta B} + 8(J-B)^2 e^{-2\beta B}}{2\beta J}$$

$$C_B = N k_B \beta^2 \frac{16B^2 e^{2\beta J} + 8(J+B)^2 e^{2\beta B} + 8(J-B)^2 e^{-2\beta B}}{\left(e^{\beta(J+2B)} + e^{\beta(J-2B)} + 2e^{-\beta J} \right)^2}$$

$$c) \quad M = Nd \frac{\partial}{\partial (\beta \mu)} \ln Z_{1-\text{dim}}$$

$$= \frac{Nd}{\beta} \frac{1}{Z_{1-\text{dim}}} \frac{\partial Z_{1-\text{dim}}}{\partial B}$$

$$\frac{\partial Z_{1-\text{dim}}}{\partial B} = 2\beta \left(e^{\beta J + 2B} - e^{\beta J - 2B} \right)$$

$$= 2\beta e^{\beta J} \left(e^{2\beta B} - e^{-2\beta B} \right)$$

$$= 4\beta e^{\beta J} \sinh(2\beta B)$$

$$M = 4Nd \frac{e^{\beta J} \sinh(2\beta B)}{e^{\beta J + 2B} + e^{\beta J - 2B} + 2e^{-\beta J}}$$

Per spin: Divide by $2Nd$

Can be compactified a little:

$$M = 2Nd \cdot \left(\frac{e^{\beta J} \sinh(2\beta B)}{e^{\beta J} \cosh(2\beta B) + e^{-\beta J}} \right)$$

Per spin: Divide by $2Nd$

$$M = 2N\alpha \frac{L_1}{L_2}$$

$$L_1 = e^{\beta J} \sinh(2\beta B)$$

$$L_2 = e^{\beta J} \cosh(2\beta B) + e^{-\beta J}$$

$$\chi = \left(\frac{\partial M}{\partial B} \right)_T = \beta \left(\frac{\partial M}{\partial \beta B} \right)_T$$

$$= 2N\alpha \beta \frac{\partial}{\partial (\beta B)} \left(\frac{L_1}{L_2} \right)$$

$$\text{Define } \frac{\partial \chi}{\partial (\beta B)} = \chi'$$

$$\chi = 2N\alpha \beta \left(\frac{L_1' L_2 - L_1 L_2'}{L_2^2} \right)$$

$$L_1' = 2 e^{\beta J} \cosh(2\beta B)$$

$$L_2' = 2 e^{\beta J} \sinh(2\beta B)$$

$$L_1' L_2 - L_1 L_2'$$

$$= 2 e^{\beta J} \cosh(2\beta B) [e^{\beta J} \cosh(2\beta B) + e^{-\beta J}]$$

$$- e^{\beta J} \sinh(2\beta B) \cdot 2 e^{\beta J} \sinh(2\beta B)$$

$$= 2 e^{2\beta J} + 2 \cosh(2\beta B)$$

$$\chi = 4Nd\beta \frac{(\cosh(2\beta B) + e^{2\beta J})}{(e^{\beta J} \cosh(2\beta B) + e^{-\beta J})^2}$$

Per spin: Divide by $2Nd$

d) $J < 0$

$\beta \gg 1$

$2\beta(|J| + B)$

$$\chi \approx \frac{4Nd\beta}{2} \frac{e}{\left(\frac{1}{2} e^{2\beta B} + e^{2\beta |J|}\right)^2}$$

$|J| > B$:

$-2\beta(|J| - B)$

$$\chi \approx 2Nd\beta e$$

$\chi \rightarrow 0$, $T \rightarrow 0$. Dimers form spin-singlets with zero magnetization $\rightarrow \chi \rightarrow 0$

$B > |J|$:

$-2\beta(B - |J|)$

$$\chi \approx 8Nd\beta e$$

$\chi \rightarrow 0$, $T \rightarrow 0$: B completely aligns spins as $T \rightarrow 0$, $\Rightarrow$ no further increase in M as B increases $\Rightarrow \chi \rightarrow 0$

Problem 3

$$a) \quad Z_g = \prod_k (1 + z e^{-\beta \epsilon_k})$$

$$z = e^{\beta \mu}$$

$$\beta p \Omega = \sum_k \ln (1 + z e^{-\beta \epsilon_k})$$

$$= \sum_{\vec{k}} \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} z^l e^{-\beta \epsilon_k l}$$

$$= \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} z^l \underbrace{\sum_{\vec{k}} e^{-\beta \epsilon_k l}}_{\int_0^{\infty} dg(\epsilon) e^{-\beta \epsilon l}}$$

$$= \frac{A}{(hc)^2} \epsilon$$

$$\sum_k e^{-\beta \epsilon_k l} = \frac{A}{(hc)^2} \underbrace{\int_0^{\infty} d\epsilon \epsilon e^{-\beta l \epsilon}}_{=1}$$

$$= \left(\frac{1}{\beta l} \right)^2 \underbrace{\int_0^{\infty} dx x e^{-x}}_{=1}$$

$$\beta p A = \frac{A}{(\beta h c)^2} \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l^3} z^l$$

$$\beta p = \frac{1}{\lambda^2} \sum_{l=1}^{\infty} b_l z^l$$

$$\underline{\underline{b_l = \frac{(-1)^{l-1}}{l^3}}}$$

Fugacity coefficient

$$\underline{\underline{\lambda = \beta h c}}$$

λ : ultra-relativistic thermal de Broglie-wavelength

$$\begin{aligned} \langle N \rangle &= \frac{\partial \ln Z_g}{\partial (\beta \mu)} = z \frac{\partial \ln Z_g}{\partial z} \\ &= z \frac{\partial \beta p A}{\partial z} \end{aligned}$$

$$g = \frac{\langle N \rangle}{A} =$$

$$g = z \frac{\partial (\beta p)}{\partial z} = \underline{\underline{\frac{1}{\lambda^2} \sum_{l=1}^{\infty} l b_l z^l}}$$

$$b) \quad \beta p = \frac{1}{\lambda^2} \left(z - \frac{1}{8} z^2 + \dots \right)$$

$$g = \frac{1}{\lambda^2} \left(z - \frac{1}{4} z^2 + \dots \right)$$

$$\beta p - g = \frac{1}{\lambda^2} \frac{1}{8} z^2 + \dots$$

$$g \lambda^2 \approx z + \dots \Rightarrow z^2 = g^2 \lambda^4 + \dots$$

$$\beta p - g = \frac{1}{\lambda^2} \frac{1}{8} g^2 \lambda^4 + \dots$$

$$\beta p = g + B_2 g^2 + \dots$$

$$\underline{B_2 = \frac{1}{8} \lambda^2} > 0$$

The pressure in an ideal Fermi-gas is higher than in an ideal classical gas, due to the Pauli-principle which acts like a repulsion. Hence $B_2 > 0$. Since $\lambda \sim h$ (Planck's constant), we see that the correction to the classical ideal gas equation of state $\beta p = g$ is a quantum effect.

$$c) \quad \beta p = g \left(1 + \frac{1}{4} g \lambda^2 + \dots \right)$$

In general

$$\beta p = \frac{1}{\lambda^2} H(z)$$

$$g = \frac{1}{\lambda^2} z \frac{\partial H}{\partial z} = \frac{1}{\lambda^2} G(z)$$

$$G(z) = g \lambda^2$$

$$z = G^{-1}(g \lambda^2)$$

$$\beta p = \frac{1}{\lambda^2} H(G^{-1}(g \lambda^2))$$

$$= \frac{1}{\lambda^2} R(g \lambda^2)$$

Hence, βp is a function of $g \lambda^2$ which may be Taylor-expanded around $g \lambda^2 = 0$.

We can truncate this power-series at low order in $g \lambda^2$ when $g \lambda^2 \ll 1$

$$g \sim \frac{1}{l_0^2}$$

l_0 : Mean separation between particles

$$g \lambda^2 = \left(\frac{\lambda}{l_0} \right)^2$$

Truncate when

$$\lambda \ll l_0$$

i.e. when relativistic thermal de Broglie wavelength is much smaller than separation between particles

$$p h c \ll l_0$$

$$k_B T \gg \frac{h c}{l_0}$$

High-temp limit, when quantum effects are small.

$$\underline{d)} \quad \beta p A = \sum_k \ln (1 + e^{-\beta(\epsilon_k - \mu)})$$

$$T \rightarrow 0 \Rightarrow \beta \rightarrow \infty$$

$$\beta p A = \int d\epsilon g(\epsilon) \ln (1 + e^{-\beta(\epsilon - \mu)})$$

$$\beta \rightarrow \infty \Rightarrow \ln (1 + e^{-\beta(\epsilon - \mu)}) = 0; \quad \epsilon > \mu$$

$$= \beta(\mu - \epsilon); \quad \epsilon < \mu$$

$$\beta p A = \beta \int_0^{\mu} d\epsilon \frac{A}{(hc)^2} (\mu - \epsilon) \epsilon$$

$$p = \frac{1}{(hc)^2} \int_0^{\mu} d\epsilon \epsilon (\mu - \epsilon)$$

$$= \frac{1}{(hc)^2} \left\{ \mu \frac{\mu^2}{2} - \frac{\mu^3}{3} \right\}$$

$$= \frac{1}{(hc)^2} \frac{1}{6} \mu^3$$

$$\langle N \rangle = \frac{\partial \ln Z_0}{\partial \beta \mu} = \sum_k \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

$$\langle N \rangle = \frac{A}{(hc)^2} \int_0^{\mu} d\epsilon \epsilon$$

$$\mathcal{G} = \frac{1}{(hc)^2} \frac{\mu^2}{2}$$

$$\mu^2 = 2 (hc)^2 \mathcal{G}$$

$$\mu = \sqrt{2} hc \mathcal{G}^{1/2}$$

$$\rho = \frac{1}{(hc)^2} \frac{1}{6} 2^{3/2} (hc)^3 \mathcal{G}^{3/2}$$

$$= \frac{\sqrt{2}}{3} hc \mathcal{G}^{3/2}$$
