

Problem 1

a) $Z_\gamma = \sum_{N=0}^{\infty} \frac{e^{\beta \mu N}}{N! h^{3N}} \cdot I_\gamma \cdot Q_N$

$$I_\gamma = \int d^3 p_1 \cdots d^3 p_N e^{-\frac{\beta}{2m} \sum_{i=1}^N \vec{p}_i^2}$$
$$= (I)^N$$

$$I = \int d^3 p e^{-\frac{\beta \vec{p}^2}{2m}} = I_0^3$$

$$I_0 = \int_{-\infty}^{\infty} dp_x e^{-\frac{\beta p_x^2}{2m}} = \sqrt{\frac{2\pi m}{\beta}}$$

$$Z_\gamma = \sum_{N=0}^{\infty} \frac{e^{\beta \mu N}}{N! h^{3N}} \left(\frac{2\pi m}{\beta} \right)^{\frac{3N}{2}} Q_N$$

Introducing $\Lambda = \frac{h}{\sqrt{2\pi m k_B T}} \Rightarrow$

$$Z_\gamma = \sum_{N=0}^{\infty} \frac{e^{\beta \mu N}}{\Lambda^{3N}} \frac{1}{N!} Q_N = \sum_{N=0}^{\infty} \frac{Z^N}{N!} Q_N$$

$$\underline{\underline{Z = \frac{e^{\beta \mu}}{\Lambda^3}}}$$

$$\underline{b)} \quad Z_g = e^{\beta p V}$$

$$\beta p V = \ln Z_g$$

$$\langle N \rangle = \frac{\partial \ln Z_g}{\partial \beta \mu} = z \frac{\partial \ln Z_g}{\partial z} = z \frac{\partial (\beta p V)}{\partial z}$$

$$Z_g = \underbrace{1 + z Q_1 + \frac{z^2}{2} Q_2 + \dots}_{\equiv x}$$

$$\beta p V = \ln(1+x) = x - \frac{x^2}{2} + \dots$$

$$\beta p V = z Q_1 + \frac{z^2}{2} Q_2 - \frac{1}{2} z^2 Q_1^2 + \dots$$

$$= z Q_1 + \frac{1}{2} (Q_2 - Q_1^2) z^2 + \dots$$

$$\langle N \rangle = z Q_1 + (Q_2 - Q_1^2) z^2$$

Defining

$$\underline{A_1 \equiv Q_1 / N}$$

$$\underline{A_2 \equiv \frac{1}{2N} (Q_2 - Q_1^2)}$$

$$\beta p = A_1 z + A_2 z^2 + \dots$$

$$g = A_1 z + 2 A_2 z^2 + \dots$$

c) Bornel virial expansion

$$\beta p = g + \sum_{l=2}^{\infty} B_l(T) g^l$$

First term: Ideal classical gas

Second term: Corrections to ideal classical gas equation of state, due to interactions between particles.

$$V_q(\vec{r}_1, \dots, \vec{r}_N) = \sum_{i=1}^N \phi(\vec{r}_i)$$

$$Q_N = \int d^3r_1 \dots \int d^3r_N e^{-\beta(\phi(\vec{r}_1) + \dots + \phi(\vec{r}_N))}$$

$$= \int d^3r_1 \dots \int d^3r_N e^{-\beta\phi(\vec{r}_1)} \dots e^{-\beta\phi(\vec{r}_N)}$$

$$= \left(\int d^3r e^{-\beta\phi(\vec{r})} \right)^N = Q_1^N$$

$$Z_g = \sum_{N=0}^{\infty} \frac{z^N}{N!} Q_N = \sum_{N=0}^{\infty} \frac{(z Q_1)^N}{N!} = e^{z Q_1}$$

$$\underline{\beta p V = z Q_1}$$

$$\langle N \rangle = z \frac{\partial \ln Z_g}{\partial z} = z Q_1$$

$$\underline{\beta p V = \langle N \rangle} \Rightarrow p V = \langle N \rangle k_B T$$

$$\Rightarrow \underline{\beta p = g} \Rightarrow \underline{B_l(T) = 0, l \geq 2}$$

All virial coefficients $B_l(T)$, $l \geq 2$ vanish.

This is so, since $V_p = \sum_i \phi(\vec{r}_i)$ does not include forces between the particles in the gas. This particular V_p only describes external forces acting on the particles.

Problem 2

$$H = - \vec{B} \cdot \sum_i \vec{S}_i$$

$$= - B \sum_i \sigma_i \quad ; \quad \sigma_i \in (-m, \dots, +m)$$

a) $Z = (Z_1)^N$

since the spins are independent

$$Z_1 = \sum_{\sigma=-m}^m e^{\beta B \sigma}$$

Introduce $\omega = \beta B$

Using the given formula on the formula sheet, we have

$$Z_1 = \frac{e^{\omega(m+\frac{1}{2})} - e^{-\omega(m+\frac{1}{2})}}{e^{\frac{\omega}{2}} - e^{-\frac{\omega}{2}}}$$

$$= \frac{\sinh(\omega(m+\frac{1}{2}))}{\sinh \frac{\omega}{2}}$$

$$\underline{Z = (Z_1)^N}$$

$$b) \langle H \rangle = \frac{1}{Z} \sum_{\{\sigma_i\}} H e^{-\beta H}$$

$$= - \frac{\partial \ln Z}{\partial \beta} = -B \frac{\partial \ln Z}{\partial \beta B}$$

$$M = \frac{1}{Z} \sum_{\{\sigma_i\}} \left(\sum_i \sigma_i \right) e^{-\beta B \sum_i \sigma_i}$$

$$= \frac{\partial \ln Z}{\partial \beta B}$$

We see that $\langle H \rangle = -BM$
 so it suffices to compute one
 of these quantities.

$$M = \frac{\partial \ln Z}{\partial \omega} = \frac{N}{Z_1} \frac{\partial Z_1}{\partial \omega}$$

$$= \frac{N}{Z_1} \left\{ \frac{(m + \frac{1}{2}) \cosh(\omega(m + \frac{1}{2}))}{\sinh(\frac{\omega}{2})} \right.$$

$$\left. - \frac{1}{2} \frac{\cosh(\frac{\omega}{2}) \cdot \sinh(\omega(m + \frac{1}{2}))}{\sinh^2(\frac{\omega}{2})} \right\}$$

$$\underline{M = N \left\{ \left(m + \frac{1}{2}\right) \coth\left(\frac{\omega}{m + \frac{1}{2}}\right) - \frac{1}{2} \coth\left(\frac{\omega}{2}\right) \right\}}$$

$$\underline{\langle H \rangle = - BM}$$

$$\underline{\omega \gg 1}: \quad \coth\left(\omega\left(m + \frac{1}{2}\right)\right) \rightarrow 1$$

$$\coth\left(\frac{\omega}{2}\right) \rightarrow 1$$

$$M = N \left\{ m + \frac{1}{2} - \frac{1}{2} \right\} = \underline{Nm}$$

Fully spin-polarized system.

$$\underline{\omega \ll 1}: \quad \coth x \approx \frac{1}{x} + \frac{x}{3}$$

$$M = N \left\{ \left(m + \frac{1}{2}\right) \left\{ \frac{1}{\omega\left(m + \frac{1}{2}\right)} + \frac{\omega\left(m + \frac{1}{2}\right)}{3} + \dots \right\} \right.$$

$$\left. - \frac{1}{2} \left(\frac{1}{\frac{\omega}{2}} + \frac{1}{3} \frac{\omega}{2} + \dots \right) \right\}$$

$$= \frac{N}{3} \omega \left[\left(m + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \right]$$

$$= \frac{N\omega}{3} m(m+1) = \frac{N}{3} m(m+1) \frac{B}{k_B T}$$

Magnetization vanishes at very high T due to thermal fluctuations.

$$c) \quad \chi = \left(\frac{\partial M}{\partial B} \right)_T = \beta \left(\frac{\partial M}{\partial \omega} \right)_T$$

$$= N\beta \left[\left(\frac{1/2}{\sinh(\frac{\omega}{2})} \right)^2 - \left(\frac{(m+1/2)}{\sinh(\omega(m+1/2))} \right)^2 \right]$$

d)

$\omega \gg 1$: First term dominates

$$\chi \approx N\beta e^{-\beta B} \xrightarrow{T \rightarrow 0} 0$$

At very low T , the system tends to fully ordered state, and there is no further increase in M by increasing B .

$\omega \ll 1$: Do it in two ways

i) Use result for M from b), for $\omega \ll 1$

$$M = \frac{N}{3} m(m+1) \frac{B}{k_B T}$$

$$\chi = \left(\frac{\partial M}{\partial B} \right)_T = \frac{N}{3} m(m+1) \frac{1}{k_B T}$$

ii) Use directly the general expression for χ , which is of the form

$$\chi = N\beta \left[\left(\frac{\alpha}{\sinh \alpha \omega} \right)^2 - \left(\frac{\gamma}{\sinh \gamma \omega} \right)^2 \right]$$

$$\alpha = \frac{1}{2}$$

$$\gamma = m + \frac{1}{2}$$

$$\omega \ll 1: \quad \text{Using } \sinh \alpha \omega \approx \alpha \omega + \frac{(\alpha \omega)^3}{3!} + \dots$$

$$\left(\frac{\alpha}{\sinh \alpha \omega} \right)^2 \approx \frac{1}{\omega^2} \left(1 - \frac{2}{3!} \alpha^2 \omega^2 + \dots \right)$$

$$\chi \approx N\beta \frac{1}{\omega^2} \left[1 - \frac{2}{3!} \alpha^2 \omega^2 - 1 + \frac{2}{3!} \gamma^2 \omega^2 \right]$$

$$= N\beta \frac{2}{3!} (\gamma^2 - \alpha^2) = \frac{N\beta}{3} \left[\left(m + \frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^2 \right]$$

$$= \frac{N\beta}{3} m(m+1)$$

Spin disorder more and more with increasing temperature, and the increase in magnetization by increasing the B-field therefore is reduced when increasing the temperature.

High-T limit of $Z = (Z_1)^N$:

$$Z_1 = \frac{\sinh(\omega(m + \frac{1}{2}))}{\sinh(\frac{\omega}{2})}$$

$$\omega \ll 1 \Rightarrow$$

$$Z_1 \approx \frac{\omega(m + \frac{1}{2})}{\frac{\omega}{2}} = 2m + 1$$

This is just the number of possible spin-states on each lattice site.

$$Z = (2m + 1)^N$$

is therefore just the total possible # spin states in the system

(Z is a weighted sum over all states, and when $\beta \rightarrow 0$ ($T \rightarrow \infty$) all states have the same weight = 1 $\Rightarrow$)

$$Z = (2m + 1)^N$$

Problem 3

$$a) \ln Z_g = - \sum_{\vec{k}} \ln (1 - z e^{-\beta \epsilon_k})$$

$$\langle N \rangle = z \frac{\partial \ln Z_g}{\partial z}$$

$$= \sum_{\vec{k}} \frac{z e^{-\beta \epsilon_k}}{1 - z e^{-\beta \epsilon_k}} = \sum_{\vec{k}} \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$$

$$k=0: \epsilon_k = 0$$

$$\langle N \rangle = \frac{1}{z^{-1} - 1} + \sum_{\vec{k} \neq 0} \frac{z e^{-\beta \epsilon_k}}{1 - z e^{-\beta \epsilon_k}}$$

$\underbrace{z^{-1} - 1}_{\equiv N_0}$

$$\underline{N_0 = \frac{1}{z^{-1} - 1}} \quad ; \quad z = e^{\beta \mu}$$

$$\underline{b) \quad N_0 = \frac{1}{\frac{1}{z} - 1} \Rightarrow z = \frac{1}{1 + \frac{1}{N_0}}}$$

$N_0 \gg 1$, macroscopic occupation of ground state $\Leftrightarrow$ Bose-Einstein condensation

$$z = e^{\beta \mu} \Rightarrow \mu = k_B T \ln z = -k_B T \ln \left(1 + \frac{1}{N_0}\right)$$

$\mu \approx -\frac{k_B T}{N_0} \rightarrow 0^- \quad \therefore \underline{z = 1}$

c) We proceed by computing $\langle N \rangle$

when $T \ll T_c$, i.e. the ground state is macroscopically occupied

$\mu \rightarrow 0^-$, $z = 1$ (but keep z for the moment)

$$\langle N \rangle = N_0 + \sum_{\vec{k} \neq 0} z \frac{e^{-\beta \epsilon_{\vec{k}}}}{1 - e^{-\beta \epsilon_{\vec{k}}}}$$

$$= N_0 + \sum_{\vec{k} \neq 0} \sum_{l=1}^{\infty} z^l e^{-\beta \epsilon_{\vec{k}} l}$$

$$N_0 = \sum_{l=1}^{\infty} z^l \sum_{\vec{k} \neq 0} e^{-\beta \epsilon_{\vec{k}} l} \equiv N_{D0}$$

Consider the $\vec{k}$ -summation:

$$\sum_{\vec{k} \neq 0} e^{-\beta \epsilon_{\vec{k}} l} = \frac{V}{(2\pi)^3} \int d^3 k e^{-\beta A l |\vec{k}|^{1/2}}$$

$$= \frac{V}{(2\pi)^3} 4\pi \int_0^{\infty} dk k^2 e^{-\beta A l k^{1/2}}$$

$$x \equiv \beta A l k^{1/2} \Rightarrow k = \frac{x^2}{(\beta A l)^2}$$

$$dk = \frac{2x dx}{(\beta A l)^2}$$

$$\sum_{\vec{k} \neq 0} e^{-\beta \epsilon_{\vec{k}} l}$$

$$= \frac{V}{(2\pi)^3} 8\pi \frac{1}{(\beta A L)^6} \underbrace{\int_0^\infty dx x^5 e^{-x}}_{= \Gamma(6)}$$

$$= \frac{V}{\pi^2} \frac{\Gamma(6)}{(\beta A L)^6}$$

$$\langle N \rangle = N_0 + V \frac{1}{\pi^2} \frac{\Gamma(6)}{(\beta A)^6} \underbrace{\sum_{l=1}^{\infty} \frac{l^5}{l^6}}_{\text{Li}_6(z)}$$

$T > T_c$: No not
macroscopically large $\Rightarrow z \neq 1$

$T < T_c$: No macroscopically large, $z=1$

$$\text{Li}_6(z=1) = \zeta(6)$$

$\zeta(6)$: Riemann-zeta function

$T < T_c$:

$$n = n_0 + \frac{1}{\pi^2} \left(\frac{\Gamma(6) \zeta(6)}{(\beta A)^6} \right)$$

$$\frac{n_0}{n} = 1 - \frac{1}{\pi^2} \frac{1}{n} \frac{\Gamma(6) \zeta(6)}{(\beta A)^6}$$

$$T \rightarrow T_c^-: \quad \rho_0 \rightarrow 0$$

$$(\beta A)^6 = \frac{\Gamma(6) \zeta(6)}{\pi^2 \rho}$$

$$\beta A = \left(\frac{\Gamma(6) \zeta(6)}{\pi^2 \rho} \right)^{1/6}$$

$$\text{so } T_c = A \left(\frac{\pi^2 \rho}{\Gamma(6) \zeta(6)} \right)^{1/6}$$

$$\frac{\rho_0}{\rho} \quad \text{when } T = \frac{T_c}{2} \Rightarrow \beta = 2\beta_c$$

$$\frac{\rho_0}{\rho} = 1 - \underbrace{\frac{1}{\pi^2} \frac{1}{\rho} \frac{\Gamma(6) \zeta(6)}{(\beta_c A)^6}}_{=1} \left(\frac{\beta_c}{\beta} \right)^6$$

$$= 1 - \left(\frac{T}{T_c} \right)^6 = 1 - \left(\frac{1}{2} \right)^6 = \frac{63}{64} \approx 0.98$$

d)

$$n = n_0 + C \sum_{l=1}^{\infty} \frac{z^l}{l^6} = n_0 + C \operatorname{Li}_6(z)$$

$$\beta p = C \sum_{l=1}^{\infty} \frac{z^l}{l^7} = C \operatorname{Li}_7(z)$$

$$T > T_c : n_0 = 0 \quad C = \frac{1}{\pi^2} \frac{\Gamma(6)}{(\beta A)^6}$$

$$\frac{\beta p}{n} = \frac{\operatorname{Li}_7(z)}{\operatorname{Li}_6(z)}$$

Ideal gas, classical case:

$$\frac{\beta p}{n} = 1$$

From the definition of $\operatorname{Li}_s(z)$, it is clear that $\operatorname{Li}_{s'}(z) < \operatorname{Li}_s(z)$; $s' > s$

$$\text{Thus, } \frac{\operatorname{Li}_7(z)}{\operatorname{Li}_6(z)} < 1 \quad \text{for all } z > 0$$

and hence

$$\left(\frac{\beta p}{n} \right)_{\text{ideal Bose gas}} < \left(\frac{\beta p}{n} \right)_{\text{ideal gas}}$$

$\Rightarrow$ (for given n):

$$\underline{(\beta p)_{\text{ideal Bose gas}} < (\beta p)_{\text{ideal gas}}}$$

$$T \rightarrow T_c^+ : z \rightarrow 1$$

$$\text{Lis}(z=1) = \zeta(s)$$

$\zeta(s)$: Riemann ζ -function

$$T = T_c$$

$$\frac{(\beta p)_{\text{Box}}}{(\beta p)_{\text{classical}}} = \frac{p_{\text{Box}}}{p_{\text{classical}}} = \frac{\zeta(7)}{\zeta(6)} \approx \underline{\underline{0.991}}$$