NORGES TEKNISK- NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR FYSIKK
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Eksamen TFY 4240: Elektromagnetisk teori

Thursday December 14 2006

kl. 09.00-13.00

English

Allowed help: C .

Rottmann: Matematisk Formelsamling (alle språkutgaver)
Barnett & Cronin: Mathematical Formulae
Øgrim: Størrelser og enheter i fysikken
Allowed calculator, empty memory, accordint to NTNU list
See also formulae page 7-10.

Each subsection has the same wight
Problems by:

Ola Hunderi Jon Andreas Støvneng

Problem 1

In this problem we will study the multipole expansion of the static scalar potential V(r). We assume known that at large distances the potential from a static dipole with dipole momentum $\vec{p}$, is given by:

$$V(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

a) Four charges, one with charge q, one with charge 3q and 2 with charge -2q are situated as shown in figure 1. All are at a distance a from the origin. Find a simple approximate expression for the potential, valid at points far from the origin. Express your answer in spherical coordinates.

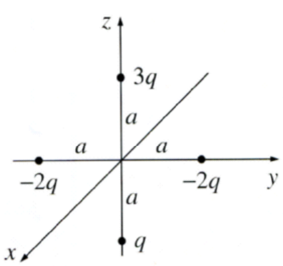

Figure 1

Given: $\vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau'$

Next, calculate the potential at points far from the origin for four charges (two dipoles) placed along the z- axis:

+q is placed at (0,0,-2a)
-q is placed at (0,0,-a)
-q is placed at i (0,0,a)
+q is placed at (0,0,2a)
Express your answer in polar coordinates. a/r is small.

Hint: Expand the exact expression for the potential and look for terms that goes as $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}$. Find in this way the monopole-, dipole- and quadrupole-contribution.

Or start with:

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} + \frac{1}{4\pi\varepsilon_0 r^3}\int r'^2\rho(\vec{r}')\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)d\tau' + ...$$

c) Two line charges are lying in the xy-plane and parallel to the x-axis at a distance a from the axis. The left carries a charge -λ per meter and has y coordinate -a , the right carries a line charge equal to +λ per meter and has y-coordinate +a. Calculate the potential far away from the linecharges. Assume that a/r is small so you can expand.

Hint: Show first that he potential at a distance r from a single line charge is given by:
$V(r) = -\frac{\lambda}{2\pi\varepsilon_0}\ln(r) + Const.$

Problem 2

a) Find the integral form of Faradays law

$$\oint \vec{E}d\vec{l} = -\frac{d}{dt}\int_S \vec{B}d\vec{A}$$

and Gauss' law for the magnetic field

$$\oint \vec{B}d\vec{A} = 0$$

starting from the corresponding differential forms. Assume that the surface S does not change with time

b) Find the general boundary conditions $\Delta E_t = 0$ *and* $\Delta B_n = 0$, ie. that the tangential component of the electric and normal component of the magnetic field are continuous everywhere at a boundary.

We will now look at standing electromagnetic waves

$$\vec{E}(x,y,z,t) = \vec{E}(x,y,z)\cdot e^{-i\omega t} \quad ; \quad \vec{B}(x,y,z,t) = \vec{B}(x,y,z)\cdot e^{-i\omega t}$$

inside a rectangular cavity as seen in the figure. The walls are perfect conductors. The cavity has dimensions a, b, c as seen in figure 2.

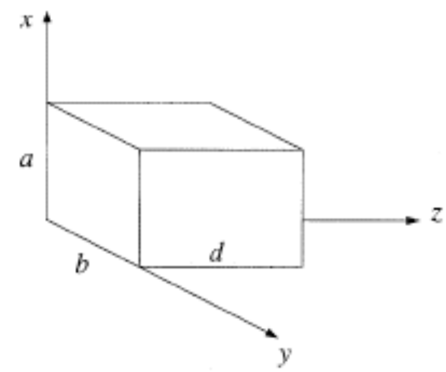

Figure 2

c) The spatial part of $\vec{E}(x,y,z,t)$ is given by

$$\begin{aligned}\vec{E}(x,y,z) = & A_x \cos k_x x \cdot \sin k_y y \cdot \sin k_z z \cdot \hat{x} + \\ & A_y \sin k_x x \cdot \cos k_y y \cdot \sin k_z z \cdot \hat{y} + \\ & A_z \sin k_x x \cdot \sin k_y y \cdot \cos k_z z \cdot \hat{z}\end{aligned}$$

where A_x, A_y and A_z are unknown coefficients. Use the wave equation $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \partial^2 \vec{E} / \partial t^2$, and the boundary conditions to determine allowed values of the angular frequency ω.

d) Determine the frequencies (f=ω/2π) of the three lowest modes in a microwave oven with metallic walls and dimensions a = 25 cm, b = 40 cm and d = 30 cm. What is $\vec{B}(x,y,z)$ for the mode with lowest frequency?

Problem 3

We shall study dipole radiation from an oscillating charge and current distribution with time variation given by $e^{i\omega t}$. The vector potential in the general case is given by

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{r_{iP}} e^{i(\omega t - k r_{iP})} d\tau'$$

See the figure3. In the equation is furthermore $r_{iP} = |r - r'| = \mathscr{r}$

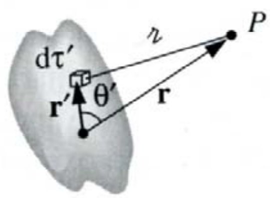

Figure 3

a) Explain why we get terms of the form $e^{i(\omega t - kr_{iP})}$ in the integrand. Explain further which condition we assume when we in the radiation zone to lowest order write the vector potential in the form:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{e^{i(\omega t - kr)}}{r}\int \vec{J}(\vec{r}\,')d\tau'$$

b) The vector potential in a) can be written as

$$\vec{A}(\vec{r},t) = i\omega\frac{\mu_0}{4\pi}\vec{p}_0\frac{e^{i(\omega t - kr)}}{r}$$

Show that this vector potential leads to a B-field given by:

$$\vec{B}(\vec{r},t) = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\sin\theta\frac{e^{i(\omega t - kr)}}{r}\hat{\phi} = \frac{\mu_0\omega^2}{4\pi c}(\hat{r}\times\vec{p}_0)\frac{e^{i(\omega t - kr)}}{r}$$

c) We shall next look at Poyntings vector. Show first that Poyntings vector in the radiation zone can be written in the form:

$$\vec{S} = \sqrt{\frac{\varepsilon_0}{\mu_0}}E^2\hat{r}$$

Explain that the factor $\sqrt{\frac{\varepsilon_0}{\mu_0}}$ has the dimension Ω^{-1} and calculate its value.

One can show that the E-and B-field in the radiation zone are related by

$$\vec{E} = -c(\hat{r}\times\vec{B}) \ ; \ \vec{B} = \frac{1}{c}(\hat{r}\times\vec{E})$$

You are not asked to show this. Given the information you now have, show that Poyntings vector in the radiation zone can be written in the form:

$$\vec{S} = Y\cdot(\hat{r}\times\vec{A})^2\hat{r}$$

Find Y.

Given: $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

d) Assume now that we have a dipole of finite length, a dipolar antenna. See figure. Explain which of the assumptions in a) which are not fulfilled in this case. The vector potential for such an antenna is in general given by

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{e^{i(\omega t - kr)}}{r}\int \vec{J}(\vec{r}\,')e^{ik\vec{r}\,'\cdot\hat{r}}d\tau'$$

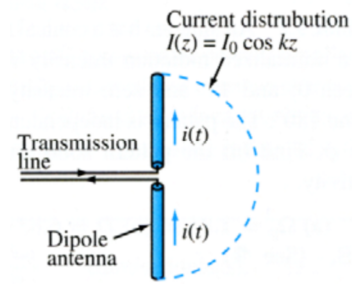

Figure 4

Assume that the dipole is a long thin antenna (see figure 4) oriented along the z-axis and where the current is given by

$$I(z) = I_o \cos kz \quad for \quad \frac{-\lambda}{4} \le z \le \frac{\lambda}{4} \quad and \;\; 0 \;\; otherwise. \quad \lambda = \frac{2\pi}{k}$$

Show that the vector potential of the antenna in the radiation zone is given by

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{2k\pi} I_0 \frac{e^{i(\omega t - kr)}}{r} \frac{\cos\left[\frac{\pi}{2}\cos(\theta)\right]}{\sin^2\theta}\hat{z}$$

and calculate the angular variation of the radiated intensity $\frac{dP}{d\Omega}$.

Given: $\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$; $\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$