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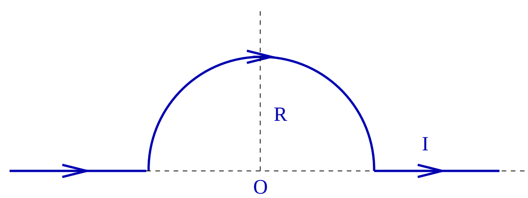
Exam in TFY4240 Electromagnetic Theory

Wednesday Dec 10, 2008
09:00–13:00

Allowed help: Alternativ **C**
Authorized calculator and mathematical formula book

This problem set consists of 8=1one page=0.

Problem 1.



An infinitely long wire carries a (time-independent) current I. The wire is bent so as to have a semi-circular detour, of radius R, around the origin O (see figure).

a) Derive an expression for the magnetic field (vector), $\mathbf{H}$, at the origin O of the coordinate system.

b) Determine the numeric value of this magnetic field given the current $I = 1\text{A}$ and radius $R = 1\text{cm}$.

Problem 2.

In this problem, we will consider the so-called *attenuation constant* for a plane wave propagating in a good conductor. We aim at, step-by-step, to derive an expression for this constant.

The medium under study is an ohmic conductor of permittivity ε, permeability μ and conductivity σ. For simplicity these constants are assumed to be *independent* of frequency.

a) From the Maxwell's equations and Ohm's law, show that the relevant wave equation reads

$$\nabla^2\mathbf{E} - \mu\varepsilon\partial_t^2\mathbf{E} - \mu\sigma\partial_t\mathbf{E} = 0, \tag{1}$$

where $\mathbf{E} \equiv \mathbf{E}(\mathbf{r}, t)$ and $\partial_t = \frac{\partial}{\partial t}$.

b) For a wave of angular frequency ω, $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r})e^{-i\omega t}$, Eq. (1) can be written in the from

$$\nabla^2\mathbf{E}_0 + \mu\epsilon(\omega)\omega^2\mathbf{E}_0 \quad = \quad 0. \tag{2}$$

Show this, and identify the function $\epsilon(\omega)$ (different from ε).

A plane wave is incident on the conductor along the inward normal, whose direction is taken to be the z-direction. Then in the conductor the electromagnetic wave can be represented by

$$\mathbf{E} \quad = \quad \mathbf{E}_0\, e^{ikz-i\omega t}, \tag{3}$$

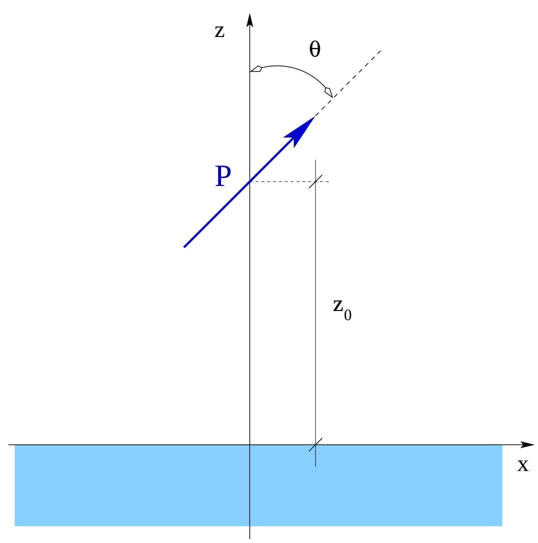
where k is the wave number.

c) Find an expression for the wave number k in terms of ω and the medium parameters (ε, μ and σ).

d) We write $k = k_1 + ik_2$, where k_1 and k_2 both are real functions. For a good conductor, *i.e.* for $\sigma/(\varepsilon\omega) \gg 1$, show that $k_1 = k_2$ and determine this common function (again) in terms of ω and the material parameters.

e) Argue why it is reasonable to name the constant $\delta = 1/k_2$ the *attenuation constant*. Write down the expression for this constant (δ).

Problem 3.



A static electric dipole is located in vacuum at position $\mathbf{r}_0 = (0, 0, z_0)$ (see figure). Its dipole moment can be written $\mathbf{p} = p(\sin\theta, 0, \cos\theta)$ where θ is the angle between $\mathbf{p}$ and the positive z-axis. Initially vacuum is filling the whole space (also the region $z \leq 0$).

a) Show that the scalar potential for an individual dipole (without the conducting half-space present) can be written as

$$V(\mathbf{r}) \quad = \quad \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{R}}\cdot\mathbf{p}}{R^2}. \tag{4}$$

What is the meaning of $\mathbf{R}$ in this equation? In your proof, you may for simplicity set $z_0 = 0$ and $\theta = 0$. Below, however, this assumption will *not* be made.

Now a perfectly conducting, *grounded*, half-space is placed at $z \leq 0$.

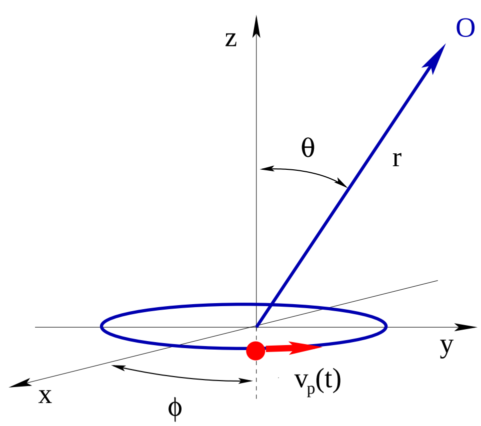
b) Give the boundary conditions that the scalar potential, $V(\mathbf{r})$, satisfies at the interface of the metallic half-space ($z = 0$). Explain (in words) the essence of the method of images.

c) Use the results from point b) to determine the location and orientation of the image dipole, and make a sketch of the resulting configuration. Moreover, show that the scalar potential for $z \geq 0$ can be written as

$$V(\mathbf{r}) \;=\; \frac{p}{4\pi\varepsilon_0}\left(\frac{x\sin\theta + (z-z_0)\cos\theta}{[x^2+y^2+(z-z_0)^2]^{3/2}} + \frac{-x\sin\theta + (z+z_0)\cos\theta}{[x^2+y^2+(z+z_0)^2]^{3/2}}\right). \tag{5}$$

d) Determine the induced (surface) charge density, $\sigma(x, y)$ on the surface of the metal. Express your answer in terms of the spatial coordinates x and y, the dipole height z_0, the dipole orientation θ, and the magnitude of the dipole moment $|\mathbf{p}| = p$.

Problem 4.



Consider a particle of charge $q \neq 0$ that is moving with a constant angular frequency ω along a circular path of radius r_0 in the xy-plane (see figure). For instance, this can be achieved by applying a static magnetic field $\mathbf{H}$. It is assumed that the particle velocity is *non-relativistic* ($v_p \ll c$).

An observer point, O, is defined by the spherical coordinates (r, θ, ϕ) relative to a coordinates system with origin in the centered of the circle (see figure).

a) Write down an expression for the *time-dependent* particle position, $\mathbf{r}_p(t)$, and use this to calculate the particle velocity, $\mathbf{v}_p(t)$, and acceleration, $\mathbf{a}_p(t)$. What is the direction of the applied (static) magnetic field, $\mathbf{H}$, relative $\mathbf{v}_p(t)$, for the particle to make circular motion?

We will now study the radiation from this particle. The time-dependent radiated power per solid angle is given by

$$\frac{dP}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c^3}\left(\hat{\mathbf{R}} \times \mathbf{a}_p\right)^2. \tag{6}$$

b) Explain what the time-dependent factor $\hat{\mathbf{R}}(t)$ in Eq. (6) means. When calculating $dP/d\Omega$, what time should be used for this quantity and $\mathbf{a}_p(t)$?

c) Under the assumption $r_0 \ll r$ derive an expression for the *time-averaged* radiated power per solid angle $\langle dP/d\Omega \rangle$ for the particle. Why is this expression independent of the angle ϕ? Explain why the assumption $r_0 \ll r$ simplifies significantly the calculation.

d) What is the total radiated power, P, from the system (independent of radiation direction)? Compare your result with Larmor's formula (*cf.* formula sheet).

e) You have just showed (hopefully) that the particle is radiating, *i.e.* that $P \neq 0$. However, still the particle performs circular motion of constant angular velocity, and therefore has time independent total energy. Explain how this is possible. Where is the radiated energy coming from?

FUNDAMENTAL CONSTANTS

ϵ_0	=	$8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	=	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	=	$3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	=	$1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	=	$9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \dfrac{1}{2}\displaystyle\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \dfrac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \dfrac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$