NTNU **Institutt for fysikk**

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Exam in TFY4240 Electromagnetic Theory

Wednesday Dec 9, 2009

09:00–13:00

Allowed help: Alternativ **C**

Standard calculator

K. Rottman: *Matematisk formelsamling* (all languages).

Formula appendix, 4 pages.

This problem set consists of 5 pages.

Each of the 16 subproblems in this exam (1a, 1b ..., 2a, ...) counts 5 points each, for a total of 80 points. The total number of points on the mid-term project was 20 points. The following table recommended by NTNU will be used to convert your total scores (mid-term + written exam) to the A–F scale:

Grade	Points
A	100–90
B	89–80
C	79–60
D	59–50
E	49–40
F	39–0

Good luck!

Problem 1. Potential theory

We are now going to investigate some properties of the electromagnetic scalar and vector potentials, $V \equiv V(\mathbf{r}, t)$ and $\mathbf{A} \equiv \mathbf{A}(\mathbf{r}, t)$, respectively. For simplicity, you can assume that $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$.

a) State how the scalar potential, V, and the vector potential, $\mathbf{A}$, are related to the electric field, $\mathbf{E}$, and the magnetic field, $\mathbf{B}$. Using these expressions, explain whether V and $\mathbf{A}$ are uniquely defined or not. Justify your answer.

b) With Gauss' law as a starting point, show that the wave equation for the scalar potential, V, reads

$$\nabla^2 V - \frac{1}{c^2}\partial_t^2 V = -\frac{\rho}{\varepsilon_0} - \partial_t \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\partial_t V \right). \tag{1}$$

In Equation (1), $c = (\varepsilon_0 \mu_0)^{-1/2}$ is the speed of light in vacuum, ρ is the electric charge density, ε_0 is the vacuum permittivity, and $\partial_t \equiv \frac{\partial}{\partial t}$.

c) We can choose our potentials so that the following condition, known as the Lorentz gauge condition, holds:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\partial_t V = 0.$$

(You do not need to prove this!) Determine the simplest possible form of the wave equation for V and $\mathbf{A}$ in this gauge. (Hint: Derive a wave equation for $\mathbf{A}$ first.)

Problem 2. The image method

In this problem, we are going to investigate two configurations of the image problem. In both problems, we have a point charge q "stuck" at the origin of our coordinate system, as depicted in Figure 1. Furthermore, we assume rotational symmetry around the x_3 axis. In all the subproblems, you can assume that a is a positive constant with dimension length.

a) First, explain *briefly* the method of images. Second, assume that the conducting half-space at $x_3 = -a$ in Figure 1(a) is grounded. State explicitly the two boundary conditions for the scalar potential, V, in this case.

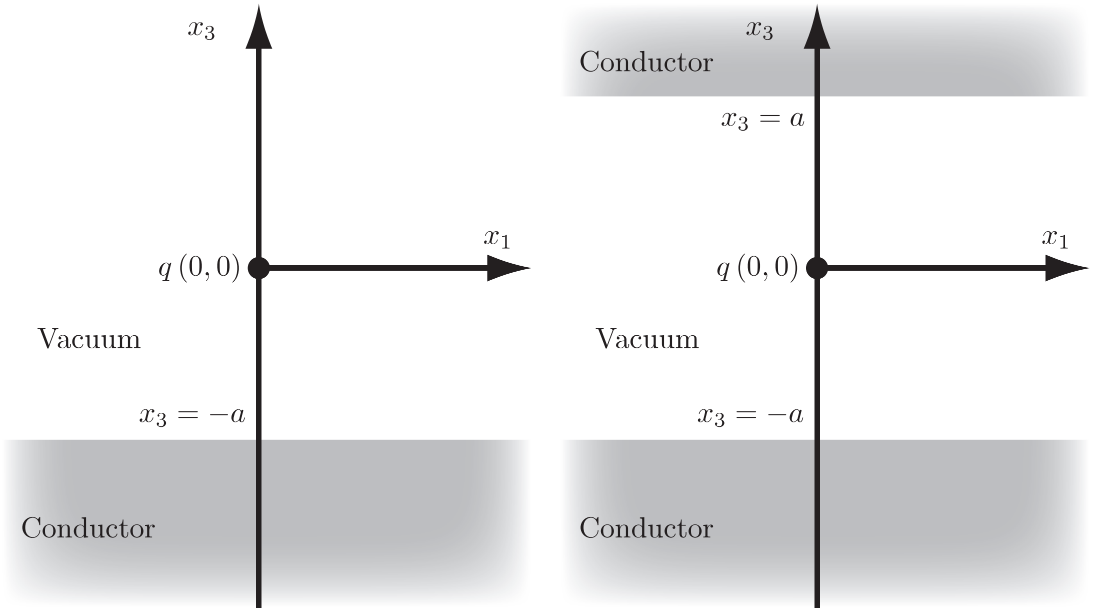
b) Solve the classic image problem where we consider a point-like particle with charge q held at the origin, "hovering" over an infinite conducting half-space at $x_3 = -a$, as depicted in Figure 1(a). Sketch the equivalent image problem, give your solution for V, and show that it fulfills one of the boundary conditions found in Problem **a)**. You are reminded that the potential from a point charge reads

$$V = \frac{q}{4\pi\varepsilon_0 r}.$$

where, in this context, r is the distance from the point charge to the observation point.

c) Now, introduce a second infinite conducting half-space at $x_3 = a$, as shown in Figure 1(b). Sketch and describe your setup of image charges in this case. Note that to solve this problem, you need to take account of both the real charge and the image charge introduced in **b)**. (Hint: you need *a lot* of image charges, so be systematic!)

d) How would you proceed to determine the induced charge density on the two grounded planes in **c)**? You do not need to perform the actual calculation.



(a) Image problem with one interface. (b) Image problem with two interfaces.

Figure 1: The two image problems. To the left is the "classic" image problem for a point charge outside an infinite conducting half-space. To the right, we have introduced an additional infinite conducting half-space.

Problem 3. TEM modes in coaxial waveguides

In the lectures, we concluded that straight, hollow metallic waveguides do not support Transverse Electric Magnetic (TEM) propagation modes. The proof rested on the fact that the inside of the waveguide was hollow. We will now show that in a *coaxial* metallic waveguide, depicted in Figure 2, we *can* have TEM modes, as this waveguide is not hollow.

In the whole problem, you can assume that the metal in the core and shield are perfect, meaning that the conductivity, σ, is infinite. We make use of cylindrical coordinates, with r, ϕ being the coordinates in the plane perpendicular to the cable, and z being the coordinate along the length of the cable. The origin of our coordinate system is, for convenience, chosen to be on the center line of the core, as depicted in Figure 2. In addition, you can assume that the dielectric insulator has the constitutive relations $\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H}$.

a) Explain (briefly, in words) why the electric field near the surface of a perfect conductor has no field component parallel to the surface. Also explain why the electric field has to be zero inside the perfect conductor.

b) Using Faraday's law, explain (briefly) why the normal component of the magnetic flux density, $\mathbf{B}$, must vanish at the surface of a perfect conductor.

c) Let us assume that the electromagnetic field in the region between the core and the shield can be expressed in the following form:

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \hat{\mathbf{r}}E_r(r)\mathrm{e}^{\mathrm{i}\beta z-\mathrm{i}\omega t},\\ \mathbf{B}(\mathbf{r},t) &= \hat{\boldsymbol{\phi}}B_r(r)\mathrm{e}^{\mathrm{i}\beta z-\mathrm{i}\omega t}.\end{aligned} \tag{2}$$

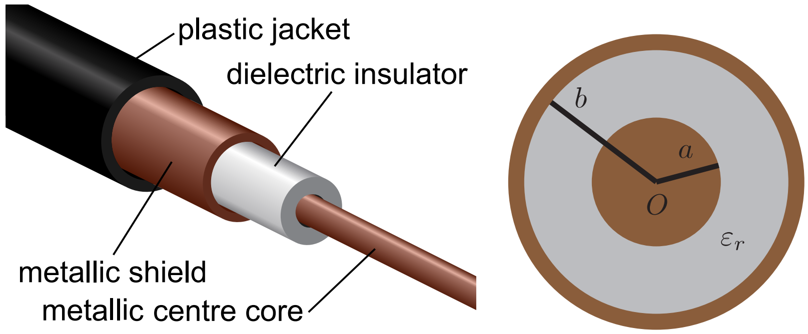


Figure 2: Explanation of names and quantities in the coaxial cable. O is the origin of the coordinate system, placed in the center of the core.

In this equation, β is the wavenumber, ω is the angular frequency of the wave, and $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{r}}$ are the unit vectors for the r and ϕ coordinates. For this form of the electric field, determine a simple way to write the derivatives ∂_ϕ, ∂_z, and ∂_t. Also show that if we let

$$\begin{aligned} E_r(r) &= \frac{aE_0}{r}, \\ B_r(r) &= \frac{\beta}{\omega} E_r(r), \end{aligned} \tag{3}$$

all four Maxwell's equations are fulfilled. Here, E_0 is an arbitrary constant with units of electric field.

d) Use these results to determine the dispersion relation for a TEM wave, i.e. Equation (2). How do the phase and group velocities depend on ε_r?

e) Determine the total (time-averaged) energy flow passing through a cross section of the coaxial cable. Verify that the units of your answer are correct.

Problem 4. EM waves in a plasma

Consider monochromatic waves in plasma (ionized gas), whose dielectric function is described by the model

$$\varepsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \tag{4}$$

where ω_p is a positive real constant known as the cutoff frequency. You are reminded that the wave equation in a source-free region reads

$$\nabla^2 \mathbf{E} - \frac{\varepsilon_r(\omega)}{c^2} \partial_t^2 \mathbf{E} = 0,$$

where we assume that the material in question is not magnetic (reasonable for a plasma). Remember that you may have to consider the cases $\omega > \omega_p$ and $\omega < \omega_p$ separately.

a) Show that the dispersion relation for the plasma can be expressed as $\omega^2 = \omega_p^2 + c^2k^2$.

b) Show that for ω below the cutoff, $\omega < \omega_p$, waves cannot propagate in the plasma.

c) Find the group velocity of waves in the plasma, $v_g(k)$.

d) Consider a linearly polarized wave of amplitude E_0 with frequency ω normally incident on a half-space region filled with plasma described by the dielectric constant in Equation (4). Find the wavenumber inside the plasma, and the amplitudes of the transmitted and reflected wave. Is your answer consistent with your results in Problem **b)**?

(Additional hint: work in the Fourier domain, or in plain English, work with monochromatic plane wave solutions to the wave equation.)

FUNDAMENTAL CONSTANTS

ϵ_0	$=$	$8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$=$	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$=$	$3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$=$	$1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$=$	$9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \frac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$