NTNU **Institutt for fysikk**

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Exam in TFY4240 Electromagnetic Theory
Dec 10, 2012
09:00–13:00

Allowed help: Alternativ **C**
Authorized calculator and mathematical formula book

This problem set consists of 4 pages.

This exam consists of three problems each containing several sub-problems. Each of the sub-problems will be given approximately equal weight during grading. However, some sub-problems may be given double weight, but only if so is indicated explicitly.

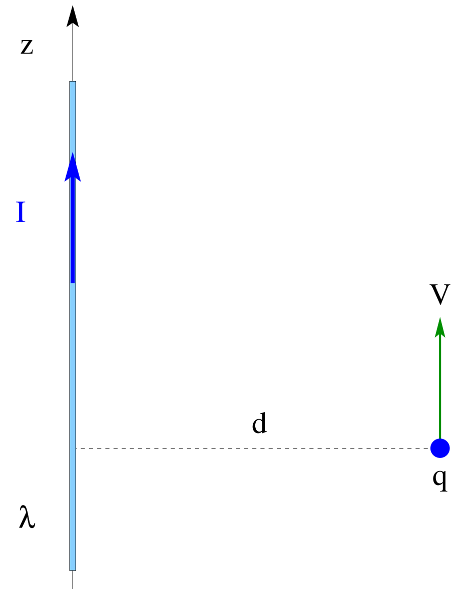
For your information, it is estimated that you will spend about 15% of the time on Problem 1; 25% on problem 2; and about 60% on Problem 3.

I will be available for questions related to the problems themselves (though not the answers!). The first round (of two), I plan to do around 10am, and the other one, about two hours later.

The problems are given in English only. Should you have any language problems related to the exam set, do not hesitate to ask. For your answers, you are free to use either English or Norwegian.

Good luck to all of you!

Problem 1.



A particle of charge q is traveling in vacuum with velocity $\boldsymbol{v}$ parallel to an infinitely long wire of a constant uniform charge distribution $\lambda > 0$ per unit length (see figure). The wire also carries a constant current I directed as shown in the figure. In this problem the gravitational force acting on the particle can be neglected.

a) Discuss and make a drawing of the forces acting on the moving particle.

b) Write down the expressions for the charge density, $\rho(\boldsymbol{r})$, and current density, $\boldsymbol{J}(\boldsymbol{r})$, of the wire and use them to obtain the electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{H}$) fields around it.

c) Obtain an expression for the critical velocity $\boldsymbol{v}_c = v_c\hat{\boldsymbol{z}}$ that the particle must have in order to travel in a straight line parallel to the wire, a distance d away from it.

Problem 2.

A static charge distribution produces a radial electric field

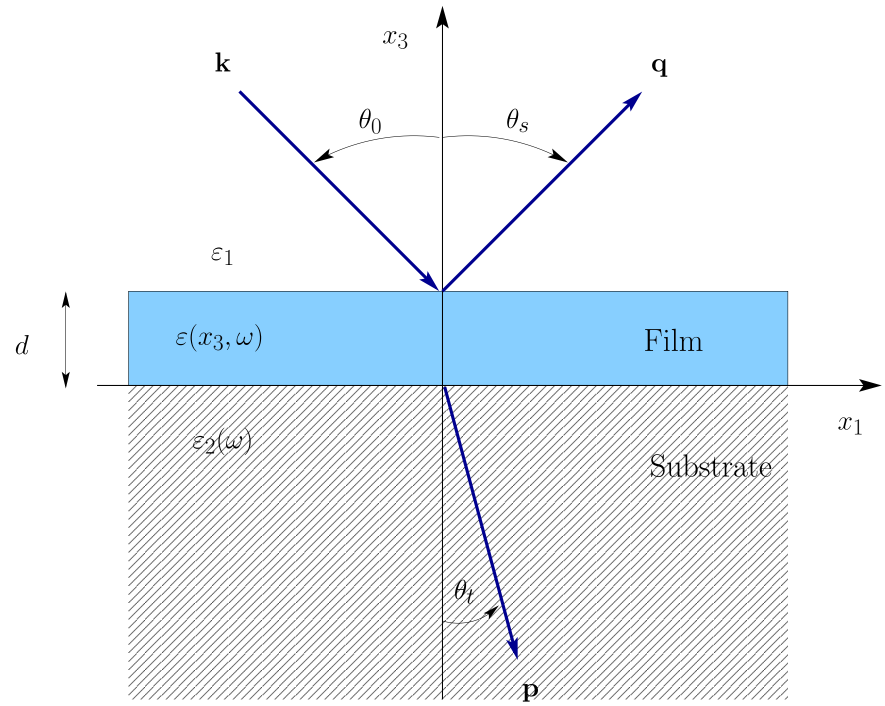
$$\boldsymbol{E}(\boldsymbol{r}) = A\frac{e^{-br}}{r^2}\hat{\boldsymbol{r}}, \tag{1}$$

where $A > 0$ and $b > 0$ are constants; r denotes the radial coordinate; and $\hat{\boldsymbol{r}}$ is a unit vector in the radial direction. Assume the medium to be vacuum.

a) (*double weight*) What is the charge density, $\rho(\boldsymbol{r})$, that produced the electric field from Eq. (1)? Make a sketch of $\rho(\boldsymbol{r})$.

b) What is the total electric charge Q?

Problem 3.



Consider the scattering geometry depicted above. Here a dielectric film is placed on top of a substrate that fills the region $x_3 < 0$. The dielectric functions of the media above and below the film are ε_1 (a constant) and $\varepsilon_2(\omega)$, respectively, where ω denotes the angular frequency. The film has thickness $d \geq 0$ and is made from a spatially local medium that is characterized by a frequency and *spatially dependent* dielectric function, $\varepsilon(x_3, \omega)$. We remind you that for a spatially local medium, the constitutive relation for the $\boldsymbol{E}$ and $\boldsymbol{D}$-fields reads $\boldsymbol{D}(\boldsymbol{x}, \omega) = \varepsilon_0 \varepsilon(x_3, \omega) \boldsymbol{E}(\boldsymbol{x}, \omega)$ (the relation is local in spatial coordinates). Notice that the spatial dependence of the dielectric function of the film only enters *via* the vertical coordinate x_3. All media are assumed to be non-magnetic, i.e. the relative permeability is $\mu = 1$ for all media. In the first part of the problem we will neglect the existence of the film, and it will only be considered in the last part.

Onto this geometry, a p polarized plane electromagnetic wave is incident in the direction of the wave vector $\boldsymbol{k}$ (see figure). A coordinate system is chosen so that the projection of $\boldsymbol{k}$ onto the $x_1 x_2$ plane is directed along $\hat{\boldsymbol{x}}_1$.

a) State the general boundary conditions that the normal and tangential components of the electromagnetic field have to satisfy at a general surface (assuming no sources at the surface).

First we assume zero film thickness, i.e. $d = 0$, so that the film does not exist.

b) Why can the magnetic field component of the electromagnetic wave above the substrate be written in the form [where $\boldsymbol{k}_\parallel \propto \hat{\boldsymbol{x}}_1$ and $\boldsymbol{x}_\parallel = (x_1, x_2, 0)$]

$$\boldsymbol{H}^{>}(\boldsymbol{x}, t) = \boldsymbol{H}^{>}(\boldsymbol{x}|\omega) \exp(-\mathrm{i}\omega t), \tag{2a}$$

where

$$\boldsymbol{H}^{>}(\boldsymbol{x}|\omega) = H_0\hat{\boldsymbol{x}}_2\left[\exp\left(\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{x}_\parallel - \mathrm{i}\alpha_1(k_\parallel)x_3\right) + r(\boldsymbol{q}_\parallel|\boldsymbol{k}_\parallel)\exp\left(\mathrm{i}\boldsymbol{q}_\parallel\cdot\boldsymbol{x}_\parallel + \mathrm{i}\alpha_1(q_\parallel)x_3\right)\right]. \tag{2b}$$

Explain the symbols $\boldsymbol{k}_\parallel$ and $\boldsymbol{q}_\parallel$ and define $\alpha_1(k_\parallel)$ so that the $\boldsymbol{H}^{>}(\boldsymbol{x},t)$ satisfies the wave equation for the magnetic field. What is the physical meaning of $r(\boldsymbol{q}_\parallel|\boldsymbol{k}_\parallel)$ and use this to identify what part of Eq. (2b) corresponds to the reflected and incident wave.

In the substrate, the magnetic field takes the form

$$\boldsymbol{H}^{<}(\boldsymbol{x}|\omega) = H_0\hat{\boldsymbol{x}}_2\, t(\boldsymbol{p}_\parallel|\boldsymbol{k}_\parallel)\exp\left(\mathrm{i}\boldsymbol{p}_\parallel\cdot\boldsymbol{x}_\parallel - \mathrm{i}\alpha_2(p_\parallel)x_3\right) \tag{3}$$

where $t(\boldsymbol{p}_\parallel|\boldsymbol{k}_\parallel)$ is the transmission amplitude.

c) (*double weight*) How is $\alpha_2(p_\parallel)$ entering Eq. (3) defined? Derive expressions for the electric fields above and below the substrate, denoted $\boldsymbol{E}^{>}(\boldsymbol{x}|\omega)$ and $\boldsymbol{E}^{<}(\boldsymbol{x}|\omega)$, respectively.

d) (*double weight*) Use the boundary conditions on the electromagnetic field to obtain a relation between $\boldsymbol{k}_\parallel$, $\boldsymbol{q}_\parallel$ and $\boldsymbol{p}_\parallel$. Obtain expressions for the $r(\boldsymbol{q}_\parallel|\boldsymbol{k}_\parallel)$ and $t(\boldsymbol{p}_\parallel|\boldsymbol{k}_\parallel)$ expressed in terms of α_i, ε_i and μ_i $(i = 1, 2)$.

e) The intensity of an electromagnetic wave crossing a plane parallel to the x_1x_2 plane is $I = |\langle\boldsymbol{S}\rangle\cdot\hat{\boldsymbol{x}}_3|$. What is the meaning of the symbols used to define the intensity, and give the SI-unit used for I. Calculate the intensity of the reflected (I_r) and transmitted (I_t) light for our scattering geometry. Express your answers in terms of r, t, α_i, ε_i, and μ_i.

The reflection (R) and transmission (T) *coefficients* (not to be confused with amplitudes r and t) are defined as

$$R = \frac{I_r}{I_0}, \qquad T = \frac{I_t}{I_0}, \tag{4}$$

where I_0 is the intensity of the incident light.

f) Derive the expressions for R and T. Also here express your answers in terms of r, t, α_i, ε_i, and μ_i. Under the assumption that there is no absorption of energy in the media involved, i.e. $\mathrm{Im}\varepsilon_i = 0$, demonstrate that the conservation of energy condition $R+T = 1$ is satisfied.

We will now assume a non-zero film thickness $d > 0$.

g) For this case, obtain a mathematical expression that determines the direction of the transmitted light. The expression should be valid for a general form of $\varepsilon(x_3, \omega)$.

h) In your own words, describe a method that can be used to determining the intensity of the transmitted light (and not only its direction) when $d \neq 0$.

FUNDAMENTAL CONSTANTS

ϵ_0	$=$	$8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$=$	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$=$	$3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$=$	$1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$=$	$9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \frac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$