NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Physics

Examination paper for TFY4240 Electromagnetic theory

Academic contact during examination: Associate Professor John Ove Fjærestad

Phone: 97 94 00 36

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Examination time (from-to): 9-13

Permitted examination support material: C

Approved calculator

Rottmann: Matematisk Formelsamling (or an equivalent book of mathematical formulas)

Other information:

This exam consists of three problems, each containing several subproblems. In many cases it is possible to solve later subproblems even if earlier subproblems were not solved.

Under normal circumstances, each subproblem will be given approximately equal weight during grading, except subproblem 1c, which may be given a higher weight.

Some formulas can be found on the pages following the problems.

Language: English

Number of pages (including front page and attachments): 9

Checked by:

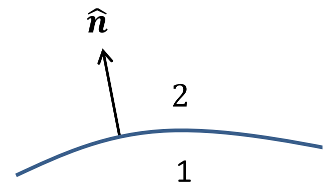
Date Signature

Problem 1.

Consider an electrostatics problem involving two dielectric media 1 and 2, with electric permittivities ϵ_1 and ϵ_2, respectively. Show that

a) inside a given medium $m = 1, 2$, the potential $V(\boldsymbol{r})$ satisfies the Poisson equation

$$\nabla^2 V = -\frac{\rho_f}{\epsilon_m}, \tag{1}$$



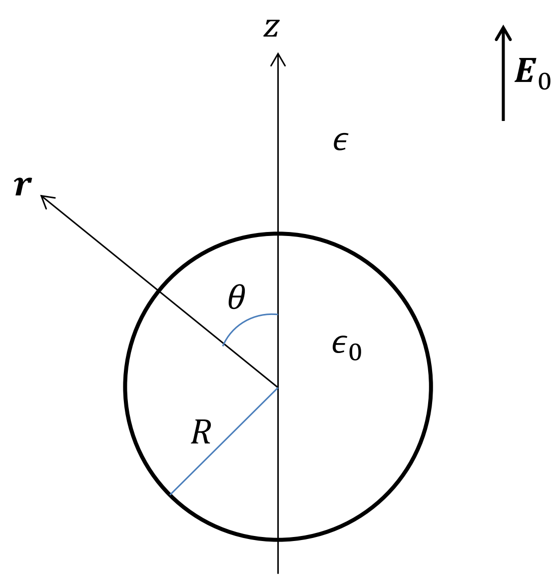
b) the boundary conditions for V at the boundary between media 1 and 2 can be written

$$V_2 - V_1 = 0, \tag{2}$$

$$\epsilon_2\,\partial_n V_2 - \epsilon_1\,\partial_n V_1 = -\sigma_f. \tag{3}$$

Here the subscripts 1 and 2 on V refer to which side of the boundary the potential is to be evaluated, and $\partial_n V \equiv \hat{\boldsymbol{n}} \cdot \nabla V$, where the unit vector $\hat{\boldsymbol{n}}$ (defined at each boundary point) is perpendicular to the boundary, pointing from medium 1 to medium 2, as shown in the figure above.

Consider (see the figure to the right) a spherical vacuum cavity[1] of radius R surrounded by a dielectric medium. The electric permittivities of the cavity and the surrounding medium are ϵ_0 and ϵ, respectively. A uniform external electric field of magnitude E_0 is imposed on the system, such that far away from the cavity the electric field $\boldsymbol{E}$ approaches the external field.



We choose a coordinate system with the origin at the center of the spherical cavity and the z axis pointing in the direction of the external field, which thus can be written $\boldsymbol{E}_0 = E_0\hat{\boldsymbol{z}}$. In the following we wish to find the potential $V(\boldsymbol{r})$ at an arbitrary point $\boldsymbol{r}$ (with spherical coordinates (r, θ, ϕ)) in the system. Due to the symmetry of the problem, $V(\boldsymbol{r})$ will be independent of ϕ and can

[1]The Norwegian word for cavity is "hulrom".

in each of the two media be expanded as

$$V(\boldsymbol{r}) = \sum_{\ell=0} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \tag{4}$$

where $P_\ell(x)$ is the Legendre polynomial of degree ℓ in the variable $x = \cos\theta$ (in particular, $P_0(x) = 1$, $P_1(x) = x$).

c) Find the potential both outside and inside the cavity. What is the electric field inside the cavity?

Problem 2.

Consider the equation

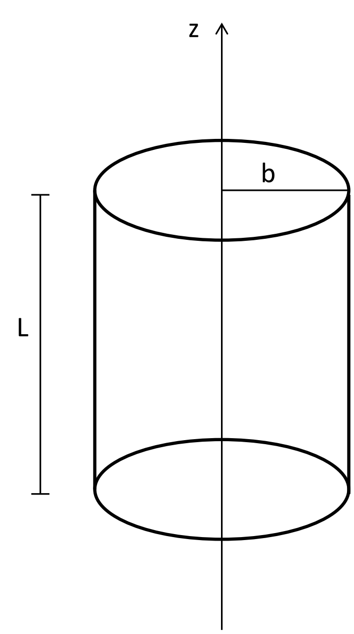
$$\frac{dU_{\text{em}}}{dt} = -\oint_a \boldsymbol{S} \cdot d\boldsymbol{a} \; - \frac{dW}{dt} \tag{5}$$

where $U_{em} = \int_\Omega d^3r \, \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2)$.

a) Explain the meaning of Eq. (5) (including the meaning of its various terms) for a general system.

A straight and infinitely long electrical wire has a circular cross section with radius b. The wire is made from an ohmic material with conductivity σ. A steady current I flows in the wire. The associated current density is uniform in the wire.

We introduce cylindrical coordinates (s, ϕ, z), with the z axis coinciding with the wire axis, such that the current flows in the positive z direction. The figure to the right shows a segment of length L of the wire.



b) Find $\boldsymbol{E}$ and $\boldsymbol{B}$ at an arbitrary point in the wire, expressed in terms of parameters given.

c) Use Eq. (5) to find an expression for dW/dt for the segment of the wire shown in the figure. How is the result related to the electrical resistance of the segment?

Problem 3.

Consider a spherical shell with a time-dependent radius $R(t)$. (As a concrete example, $R(t)$ may describe a harmonic oscillation around an average radius. However, we will not assume any particular form of the function $R(t)$ here.) The shell has a total charge Q that is conserved and at all times uniformly distributed on the shell surface. There is vacuum both outside and inside the shell. We introduce a coordinate system whose origin ($\boldsymbol{r} = 0$) is the center of the shell.

A general hint: When asked below to find $\boldsymbol{E}$ or $\boldsymbol{B}$, you are not expected to find the potential(s) first, as this may here be significantly harder than instead making use of laws obeyed by the fields.

a) Show that the (volume) charge density $\rho(\boldsymbol{r}, t)$ due to the shell is

$$\rho(\boldsymbol{r}, t) = \frac{Q}{4\pi r^2}\,\delta(r - R(t)) \tag{6}$$

where $r = |\boldsymbol{r}|$ and $\delta(u)$ is the Dirac delta function.

b) Find the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ at an arbitrary point $\boldsymbol{r}$ (outside or inside the shell).

It can be shown that the current density $\boldsymbol{j}(\boldsymbol{r}, t)$ due to the shell is

$$\boldsymbol{j}(\boldsymbol{r}, t) = \frac{Q\dot{R}(t)}{4\pi r^2}\,\delta(r - R(t))\,\hat{\boldsymbol{r}} \tag{7}$$

where $\dot{R}(t) \equiv \frac{dR(t)}{dt}$ and $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$.

c) Show that the continuity equation for electric charge,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0, \tag{8}$$

is satisfied.

d) Find the magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$ at an arbitrary point $\boldsymbol{r}$.

e) Does the shell radiate? Justify your answer.

Formulas

Some formulas that you may or may not need (you should know the meaning of the symbols and possible limitations of validity):

$$\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell,\ell'} \tag{9}$$

$$\boldsymbol{j} = \sigma \boldsymbol{E} \tag{10}$$

$$\Theta(u) \equiv \begin{cases} 0 & \text{if } u < 0 \\ 1 & \text{if } u > 0 \end{cases} \tag{11}$$

$$\frac{d}{du}\Theta(u) = \delta(u) \tag{12}$$

$$\frac{d}{du}\delta(u) = -\frac{\delta(u)}{u} \tag{13}$$

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\boldsymbol{r}', t_\mathrm{r})}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{14}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\boldsymbol{j}(\boldsymbol{r}', t_\mathrm{r})}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{15}$$

FUNDAMENTAL CONSTANTS

ϵ_0	$=$	$8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$=$	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$=$	$3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$=$	$1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$=$	$9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \frac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$