

Examination paper for TFY4240 Electromagnetic theory

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Examination time (from-to): 9-13

Permitted examination support material: C

Approved calculator

Rottmann: Matematisk Formelsamling (or an equivalent book of mathematical formulas)

Other information:

This exam consists of three problems, each containing several subproblems. In many cases it is possible to solve later subproblems even if earlier subproblems were not solved.

Under normal circumstances, each subproblem will be weighted approximately equally in the marking.

Some formulas can be found on the pages following the problems.

Language: English

Number of pages (including front page and attachments): 9

Informasjon om trykking av eksamensoppgave

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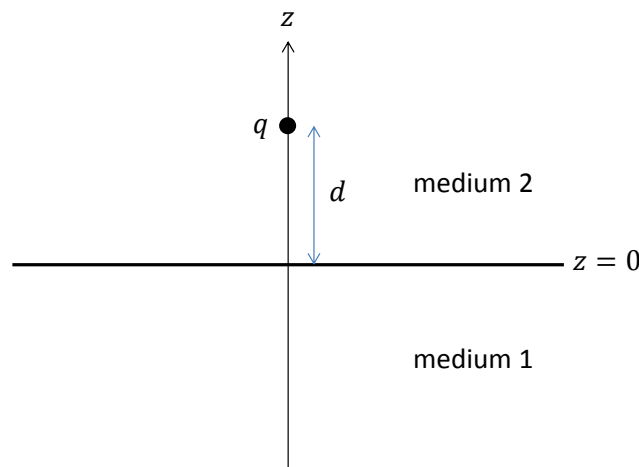
Problem 1

(a) Briefly describe the "method of images" and the type of problems it can be used to solve.

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Let space be divided into two regions filled with simple dielectric media 1 and 2 with electric permittivities ϵ_1 and ϵ_2 , respectively. The interface between the regions is an infinite plane. A point charge q sits in medium 2 a distance d from the interface.

We introduce a cartesian coordinate system with the z axis perpendicular to the interface and passing through the point charge, with origin and direction chosen such that the interface is at $z = 0$, medium 1 is in the region $z < 0$, medium 2 is in the region $z > 0$, and the point charge has coordinates $(x, y, z) = (0, 0, d)$ (see the figure).



(b) Use the method of images to find the potential V everywhere. [Hint: All image charges involved can be taken to be at distance d from the interface.]

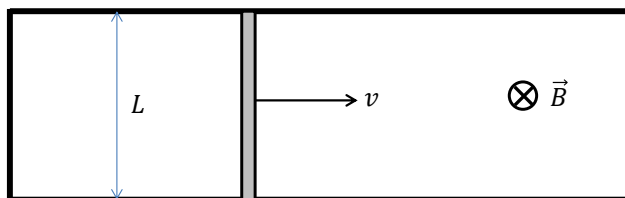
(c) Find the force \mathbf{F} on the point charge q . Comment on whether the result is reasonable for the special case $\epsilon_1 = \epsilon_2$.

(d) Find the volume bound charge density.

(e) Find the surface bound charge density.

Problem 2

As illustrated in the figure below, an ohmic bar with resistance R , length L and mass m can slide without friction on two parallel, perfectly conducting rails (which extend infinitely to the right). A uniform magnetic field \vec{B} points into the page. At time $t = 0$ a physicist sets the bar in motion by giving it an initial velocity v_0 towards the right. (For $t > 0$ the physicist does not act with any force on the bar.)



- (a) Argue that a current I will flow in the circuit consisting of the (moving) bar and the rails to the left of it. Find an expression for I which involves the bar's velocity v towards the right. What is the direction of the current?
- (b) Show that for $t > 0$ the time dependence of the current takes the form $I(t) = I_0 \exp(-\alpha t)$, and give expressions for the constants α and I_0 . [Hint: Consider Newton's 2nd law for the bar in the horizontal direction (i.e. in the bar's direction of motion).]
- (c) Calculate the total energy dissipated as Joule heat during the motion of the bar (i.e. from $t = 0$ to $t = \infty$). Give a physical interpretation of the result.

Problem 3

By introducing the scalar potential V , it can be shown that the Maxwell equations for electrostatics are equivalent to a single differential equation for V , the Poisson equation $\nabla^2 V = -\rho/\epsilon_0$, whose solution for a localized charge distribution is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (1)$$

(a) By introducing the vector potential \mathbf{A} , show that the Maxwell equations for magnetostatics are equivalent to a single differential equation for \mathbf{A} . Derive this differential equation.

(b) (i) Show that \mathbf{B} is invariant under the gauge transformation $\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\lambda$, where λ is an arbitrary function of \mathbf{r} . (ii) Argue that one can impose the so-called Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ on \mathbf{A} , and derive an expression for $\mathbf{A}(\mathbf{r})$ in the Coulomb gauge, for a (steady) localized current distribution.

(c) Show that for static problems, in the Coulomb gauge, the electromagnetic field momentum can be expressed as

$$\mathbf{P}_{\text{EM}} = \int d^3r \rho(\mathbf{r}) \mathbf{A}(\mathbf{r}). \quad (2)$$

[Hints: It may be useful to consider the cartesian components of \mathbf{P}_{EM} . You may assume that all "surface terms" at infinity (obtained from integration by parts) vanish.]

(d) Consider a system consisting of three parts: A point charge q at $(x, y, z) = (0, 0, d)$, a point charge $-q$ at $(0, 0, -d)$, and a point magnetic dipole with moment \mathbf{m} at the origin $(0, 0, 0)$. Calculate \mathbf{P}_{EM} for this system.

(e) This question concerns the roles of the Poynting vector in general problems in classical electromagnetic theory. (i) Briefly state how the Poynting vector enters into calculations of electromagnetic field momentum. (ii) The Poynting vector also appears in energy considerations. Briefly describe the interpretation/role of the Poynting vector in this context.

Formulas

Some formulas that you may or may not need (you should know the meaning of the symbols and possible limitations of validity):

$$\epsilon_2 \frac{\partial V_2}{\partial n} - \epsilon_1 \frac{\partial V_1}{\partial n} = -\sigma_f \quad (1)$$

$$\rho_b = -\nabla \cdot \mathbf{P}, \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \quad (2)$$

$$\varepsilon = -\frac{d\Phi}{dt} \quad (3)$$

$$P = UI \quad (4)$$

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k \quad (5)$$

$$\epsilon_{kij} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \quad (6)$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3} \quad (7)$$

FUNDAMENTAL CONSTANTS

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2 \quad (\text{permittivity of free space})$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2 \quad (\text{permeability of free space})$$

$$c = 3.00 \times 10^8 \text{ m/s} \quad (\text{speed of light})$$

$$e = 1.60 \times 10^{-19} \text{ C} \quad (\text{charge of the electron})$$

$$m = 9.11 \times 10^{-31} \text{ kg} \quad (\text{mass of the electron})$$

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad \begin{cases} \hat{\mathbf{x}} = \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \quad \begin{cases} \hat{\mathbf{r}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s \cos \phi \\ y = s \sin \phi \\ z = z \end{cases} \quad \begin{cases} \hat{\mathbf{x}} = \cos \phi \hat{\mathbf{s}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin \phi \hat{\mathbf{s}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \quad \begin{cases} \hat{\mathbf{s}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{array} \right.$$

In matter :

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{array} \right.$$

Auxiliary Fields

Definitions :

$$\left\{ \begin{array}{l} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \end{array} \right.$$

Linear media :

$$\left\{ \begin{array}{l} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B} \end{array} \right.$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

$$\text{Energy :} \quad U = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau$$

$$\text{Momentum :} \quad \mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B}) d\tau$$

$$\text{Poynting vector :} \quad \mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})$$

$$\text{Larmor formula :} \quad P = \frac{\mu_0}{6\pi c} q^2 a^2$$

VECTOR IDENTITIES

Triple Products

$$(1) \quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$(2) \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Product Rules

$$(3) \quad \nabla(fg) = f(\nabla g) + g(\nabla f)$$

$$(4) \quad \nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

$$(5) \quad \nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$(6) \quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$(7) \quad \nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$(8) \quad \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second Derivatives

$$(9) \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$(10) \quad \nabla \times (\nabla f) = 0$$

$$(11) \quad \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

FUNDAMENTAL THEOREMS

$$\text{Gradient Theorem : } \int_a^b (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

$$\text{Divergence Theorem : } \int (\nabla \cdot \mathbf{A}) d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$$

$$\text{Curl Theorem : } \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\tau = dx dy dz$

$$\text{Gradient :} \quad \nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence :} \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl :} \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$$

$$\text{Laplacian :} \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin \theta dr d\theta d\phi$

$$\text{Gradient :} \quad \nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\text{Divergence :} \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned} \text{Curl :} \quad \nabla \times \mathbf{v} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} \\ &+ \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \end{aligned}$$

$$\text{Laplacian :} \quad \nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$

$$\text{Gradient :} \quad \nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence :} \quad \nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl :} \quad \nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$$

$$\text{Laplacian :} \quad \nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$