NTNU **Institutt for fysikk**

Contact during the exam:
Associate Professor John Ove Fjærestad
Telephone: 97 94 00 36 or 73 59 34 48

Exam in TFY4240 Electromagnetic Theory
May 16, 2019
09:00–13:00

Allowed help: Alternativ **C**
A permitted basic calculator and a mathematical formula book (Rottmann or equivalent)

This problem set consists of 4 pages.

This exam consists of three problems, each containing several subproblems. Under normal circumstances, each subproblem (1a-3f) will be given equal weight in the grading.

I will be available for questions related to the problems themselves (though not the answers!). The first round (of two), I plan to do around 9.30 am, and the other one, about two hours later.

The problems are given in English only. Should you have any language problems related to the exam set, do not hesitate to ask. For your answers, you are free to use either English or Norwegian.

Note that some formulas are given on the pages following the last problem.

Good luck!

Problem 1.

Consider a spherical shell with radius R. We introduce spherical coordinates (r, θ, ϕ) with the origin at the center of the shell. Let the electric potential on the shell be independent of ϕ, so it can be written

$$V(R, \theta) = \sum_{\ell=0}^{\infty} V_\ell P_\ell(\cos\theta), \tag{1}$$

where the $P_\ell(\cos\theta)$ are Legendre polynomials, and the V_ℓ are expansion coefficients. There is no charge outside or inside the shell.

a) Find the potential everywhere (both outside and inside the shell).

b) Find the surface charge density on the shell.

Consider a dielectric sphere with uniform polarization $\boldsymbol{P}$. We will be interested in the electric potential produced by the polarization, and in the associated electric field.

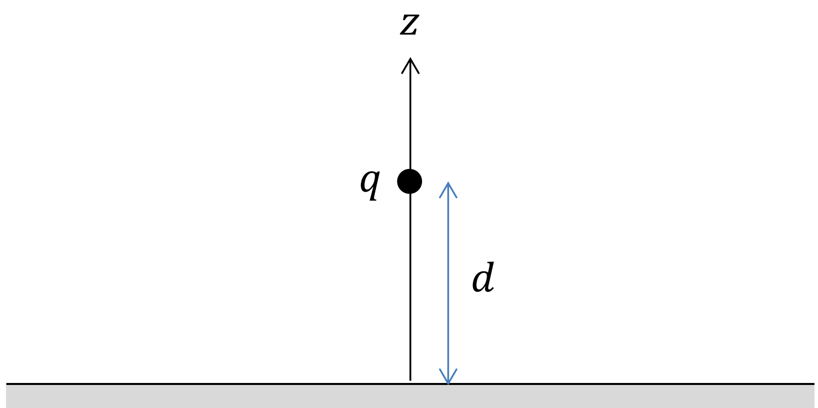
c) Find the potential and field outside the sphere.

d) Find the potential and field inside the sphere.

In both (c) and (d), comment on the form of the electric field.

Problem 2.

A point charge q is held a distance d above an infinite grounded conducting plate (see the figure). We let the xy plane coincide with the plate, and choose the origin and z axis such that the point charge has coordinates $(x, y, z) = (0, 0, d)$.



a) 1. Use the method of images to find the potential $V(x, y, z)$ above the grounded conducting plate ($z > 0$).

2. Find the force $\boldsymbol{F}_q$ on the point charge due to the conducting plate.

b) 1. Find the surface charge density $\sigma(x, y)$ on the conducting plate.

2. Find the total surface charge Q.

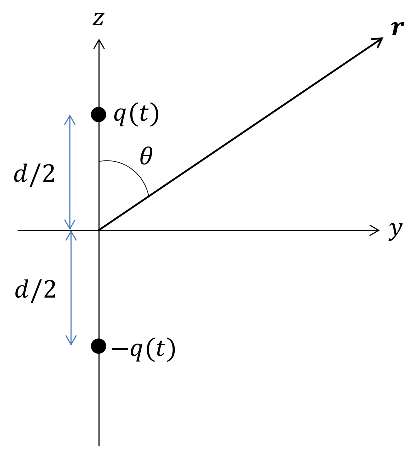
3. Write down an expression for the force $\boldsymbol{F}_q$ in terms of an integral involving $\sigma(x, y)$. Give the dependence on x and y of other factors than $\sigma(x, y)$ in the integrand (you do not need to evaluate the integral).

c) For a general problem in electromagnetism, explain the meaning of the 3 terms in the equation

$$\boldsymbol{F} = \oint_a \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} - \frac{d}{dt} \int_\Omega \frac{\boldsymbol{S}}{c^2} \, d^3 r. \tag{2}$$

d) Use (2) to give an alternative calculation of the force $\boldsymbol{F}_q$. [Hint: Let the calculation involve a hemisphere whose radius is taken to infinity.]

Problem 3.



This problem concerns radiation from a harmonically oscillating electric dipole.

Two tiny metal spheres are separated by a distance d along the z axis (see the figure). They are connected by a fine wire, through which a current with angular frequency ω is driven, so that at time t, the charge on the upper sphere is $q(t)$ and the charge on the lower sphere is $-q(t)$, where $q(t) = q_0 \cos(\omega t)$.

We wish to find the fields at a point with spherical coordinates (r, θ, ϕ). We will only be interested in the leading parts of the fields that contribute to radiation, denoted below by $\boldsymbol{E}_{\text{rad}}$ and $\boldsymbol{B}_{\text{rad}}$. This means that in your calculations you may assume the following set of inequalities: $d \ll c/\omega \ll r$. (Alternatively this may be expressed as $d \ll \lambda \ll r$, where $\lambda = 2\pi/k$ is the wavelength associated with the wavevector $k \equiv \omega/c$.) You may find it helpful to carry out some calculations using "complexified" versions of the various quantities, but your final answers should be real quantities.

a) The current density is given by

$$\boldsymbol{j}(\boldsymbol{r}, t) = \hat{\boldsymbol{z}} \frac{dq(t)}{dt} \delta(x)\delta(y)\Theta(d/2 - |z|). \tag{3}$$

(Here Θ is the (Heaviside) step function.) Justify this expression (e.g. by showing/arguing that it has/leads to various properties that should be satisfied based on the problem description.)

b) Find the vector potential ($\boldsymbol{A}_{\text{rad}}(\boldsymbol{r}, t)$).

c) Find $\boldsymbol{B}_{\text{rad}}(\boldsymbol{r}, t)$.

d) Find $\boldsymbol{E}_{\text{rad}}(\boldsymbol{r}, t)$. [Hint: Use (c) and one of Maxwell's equations.]

e) $\boldsymbol{E}_{\rm rad}$ and $\boldsymbol{B}_{\rm rad}$ can be expressed in the coordinate-free form

$$\boldsymbol{E}_{\rm rad}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi r}[\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}})], \quad \boldsymbol{B}_{\rm rad}(\boldsymbol{r},t) = -\frac{\mu_0}{4\pi rc}[\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}], \tag{4}$$

where $\ddot{\boldsymbol{p}}$ is evaluated at time $t - r/c$ (the notation $\ddot{\ }$ means the second derivative with respect to time). Using either these expressions, or your own from (c) and (d), discuss (i) the relative magnitude of $\boldsymbol{E}_{\rm rad}$ and $\boldsymbol{B}_{\rm rad}$, (ii) their directions relative to each other, (iii) their directions relative to the direction of propagation $\hat{\boldsymbol{r}}$.

f) The time-averaged radiated power of the dipole can be written

$$\langle P \rangle = \int \left\langle \frac{dP}{d\Omega} \right\rangle d\Omega, \tag{5}$$

where $\langle \frac{dP}{d\Omega} \rangle$ is the time-averaged radiated power per unit solid angle (in these expressions, the brackets $\langle \ldots \rangle$ denote the time average).

1. Find $\langle \frac{dP}{d\Omega} \rangle$. What are the directions in which the radiation is maximal and minimal?
2. Find $\langle P \rangle$.

Formulas

Some formulas that you may or may not need (you should know the meaning of the symbols and possible limitations of validity):

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \tag{1}$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx} \right)^\ell (x^2-1)^\ell \tag{2}$$

$$\text{(so } P_0(x)=1,\ P_1(x)=x,\ P_2(x)=(3x^2-1)/2, \text{ etc.)}$$

$$\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1} \delta_{\ell,\ell'} \tag{3}$$

$$\sigma = -\epsilon_0 \left[\frac{\partial V}{\partial n}\Big|_{\text{outside}} - \frac{\partial V}{\partial n}\Big|_{\text{inside}} \right] \tag{4}$$

$$\rho_b = -\nabla \cdot \boldsymbol{P}, \quad \sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} \tag{5}$$

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \tag{6}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\boldsymbol{j}(\boldsymbol{r}', t_{\text{ret}})}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{7}$$

FUNDAMENTAL CONSTANTS

ϵ_0	$= 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$= 4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$= 3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$= 1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$= 9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \frac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$