Problem 1.

Two semi-infinite grounded conducting planes (thick blue lines) meet at a right angle as shown in the left plot of Fig. 1 (the z axis (not shown) points out of the paper plane). A point charge q is held at position $(x, y, z) = (a, b, 0)$. The potential V at an arbitrary point $(x > 0, y > 0, z)$ outside the conductors can be found with the method of images by using three image charges q_1, q_2, and q_3 at the positions shown in the right plot of Fig. 1.

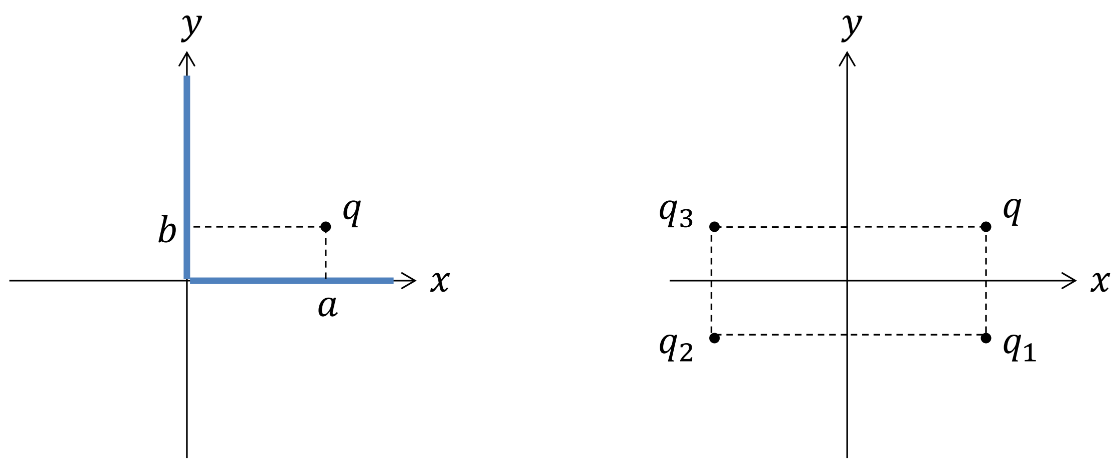

Figure 1

a)
1. Determine the image charges.
2. Find the total surface charge Q_h on the horizontal conducting plane (calculate the surface charge density first).
3. Use symmetry to find the total surface charge Q_v on the vertical conducting plane. Calculate $Q_h + Q_v$. Does the answer make sense? (Given: $\arctan(x) + \arctan(1/x) = \frac{\pi}{2} \quad (x > 0)$.)

Next, consider the system illustrated in Fig. 2 (the z axis (not shown) points out of the paper plane). A grounded conductor has a surface (thick blue line) that consists of an infinite plane with a half-sphere of radius R sticking out from it as shown in the figure. A point charge q is held at position $(x, y, z) = (a, b, 0)$.

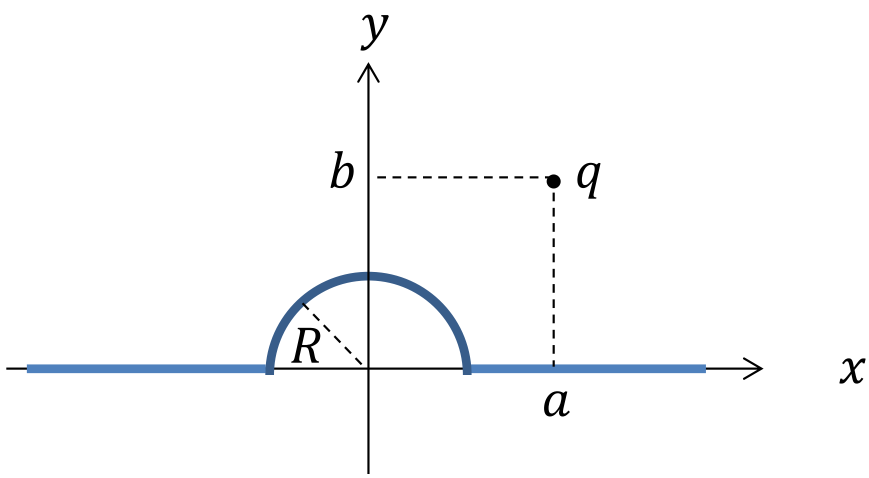

Figure 2

b) Use the method of images to find the potential V above the conductor. How many image charges are needed, and what are their positions and charges? (You may freely make use of results from simpler problems considered in the course.)

c) For positions (above the conductor) far away from the point charge q, determine the leading term in the multipole expansion of V. Find an expression for the associated multipole moment.

Problem 2.

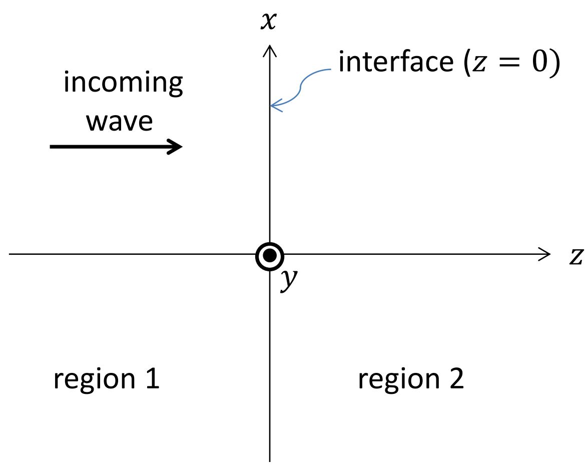

Figure 3

Consider the system in Fig. 3. Region 1 is vacuum, and region 2 is a medium described by a complex refractive index $\tilde{n} = n' + in''$, where n' and n'' are its non-negative real and imaginary parts. You may assume that $\mu = \mu_0$. Introducing the cartesian coordinate system shown, the interface between the vacuum and the medium is the xy plane ($z = 0$).

An electromagnetic monochromatic plane wave with angular frequency ω is incident on the interface from the vacuum. The electric field of the incident wave can be written

$$\tilde{\boldsymbol{E}}(\boldsymbol{r}, t) = \tilde{\boldsymbol{E}}_{0I} e^{i(k_I z - \omega t)}, \tag{1}$$

where, as usual, it is understood that the physical electric field is the real part of this expression. Here $\tilde{\boldsymbol{E}}_{0I} = \tilde{E}_{0I}\hat{x}$, and k_I is real and positive.

Remark: If you find some calculations below too difficult to carry out for $n'' \neq 0$, you may still get a significant fraction of the marks if you consider $n'' = 0$ instead.

a) Show that the complex amplitudes $\tilde{E}_{0T}$ and $\tilde{E}_{0R}$ for the transmitted and reflected waves, respectively, are given by

$$\frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2}{1+\tilde{n}}, \qquad \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{1-\tilde{n}}{1+\tilde{n}}. \tag{2}$$

b)
1. Find the intensities I_I, I_R, and I_T for the incident, reflected, and transmitted waves.
2. At the interface, find expressions for $T \equiv I_T/I_I$ and $R \equiv I_R/I_I$. Calculate $R + T$ and give an interpretation of the result.

c) (Double weight) Consider the equation

$$\boldsymbol{f} = \boldsymbol{\nabla} \cdot \overleftrightarrow{\boldsymbol{T}} - \frac{1}{c^2}\frac{\partial \boldsymbol{S}}{\partial t}. \tag{3}$$

1. For an arbitrary point in the medium, show that

$$\left\langle \frac{\partial \boldsymbol{S}}{\partial t} \right\rangle = 0 \tag{4}$$

   (as usual, the brackets denote the time average).

2. Calculate the time-averaged force $\langle \boldsymbol{F} \rangle$ on the medium, per unit area of the interface.
3. The result can be expressed in such a way that it involves I_I and R. Comment on how it depends on these two quantities; is it physically reasonable? Consider especially the cases $R = 0$ and $R = 1$.

Problem 3.

a) The electromagnetic fields at a point $\boldsymbol{r}$ at time t due to a point charge q in arbitrary motion can be written

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{R}{(\boldsymbol{R}\cdot\boldsymbol{u})^3}[(c^2 - v^2)\boldsymbol{u} + \boldsymbol{R} \times (\boldsymbol{u} \times \boldsymbol{a})], \qquad \boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c}\hat{\boldsymbol{R}} \times \boldsymbol{E}(\boldsymbol{r}, t), \tag{5}$$

where $\boldsymbol{u} = c\hat{\boldsymbol{R}} - \boldsymbol{v}$, $R = |\boldsymbol{R}|$, and $\hat{\boldsymbol{R}} = \boldsymbol{R}/R$.

1. The description given above of the expressions for the fields is not complete, as the meaning of several quantities is not given and/or is not clear. Give the missing information to make it clear precisely how the fields are defined.

Consider a point charge q that oscillates harmonically with frequency ω, such that its trajectory is $\boldsymbol{r}_q(t) = (x_q(t), y_q(t), z_q(t))$ with $x_q(t) = y_q(t) = 0$ and

$$z_q(t) = z_0 \cos \omega t, \tag{6}$$

with z_0 a constant.

2. Suppose that you were asked to numerically evaluate the electromagnetic fields due to this harmonically oscillating charge at some arbitrary point $\boldsymbol{r} = (x, y, z)$ and time t. Write down the central equation that would have to be solved.
3. How many solutions to that equation do you expect? Briefly justify your answer.

In the remainder of this problem we will be interested in some further aspects of the fields due to the point charge (6), at large distances $r \gg z_0$, in the nonrelativistic limit $v/c \ll 1$.

b) Use Eq. (5) to find expressions for the *radiation* fields at the point $\boldsymbol{r}$ at time t.

c)
1. Find the time average of the associated Poynting vector.
2. Find the energy radiated during one period of the oscillation.