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Exam in TFY4240 - Electromagnetic theory

May 22, 2025
09:00-13:00

Permitted examination support material: Alternative **C**:

> A permitted basic calculator and a mathematical formula book (K.Rottmann or equivalent).

This problem set consists of 11 pages. The four last pages constitute a formula sheet from Griffiths' Electrodynamics (fourth edition).

The number of points available for each problem is indicated next to the title. The relative weighting of the problems can be subject to change, but it nevertheless gives you an idea of how much time I estimate you will spend on each problem.

If a question is unclear/vague, make your own assumptions and specify in your answers the premises you have made.

The problems are given in English only. You are free to use either English or Norwegian for your answers.

Good luck!

This problem set was developed by Sondre Duna Lundemo.

Problem 1 - Short questions (20 points)

The sub-problems in this problem deal mostly with different systems and can be answered independently of each other. The questions should only require short derivations or short explanations.

A point dipole can be constructed in the following way: Consider two point charges, one of charge q and the other of charge $-q$, separated by some vector $\mathbf{d}$ (from $-q$ to q). The point dipole is obtained by letting $|\mathbf{d}| \to 0$, in such a way that the product $q|\mathbf{d}|$ is finite.

(a) Show that a point dipole situated at $\mathbf{r} = \mathbf{r}_0$ is described by the charge density

$$\rho(\mathbf{r}) = -\mathbf{p} \cdot \boldsymbol{\nabla}\delta(\mathbf{r} - \mathbf{r}_0), \tag{1}$$

where $\mathbf{p} := q\mathbf{d}$. **Hint**: Use a Taylor expansion.

(b) Use the charge density in Eq. (1) to derive the scalar potential $V(\mathbf{r})$ of the point dipole.

In the absence of sources, the electromagnetic momentum density $\mathbf{g} := \mathbf{D} \times \mathbf{B}$ can be shown to satisfy the equation

$$\partial_t \mathbf{g} - \boldsymbol{\nabla} \cdot \overset{\leftrightarrow}{\mathbf{T}} = 0, \tag{2}$$

where $\overset{\leftrightarrow}{\mathbf{T}} := T_{ij}\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j$ denotes the Maxwell stress tensor (summation convention used).

(c) Explain the physical meaning of Eq. (2).

Suppose we describe a metal as a gas of mobile electrons flowing through the crystal lattice of the positive ions. On macroscopic scales, the metal is overall charge neutral. When we place a fixed charge Q into this system, say at $\mathbf{r}_0$, the electron charge density $\rho(\mathbf{r})$ of the metal is altered. A simple model for the scalar potential in the metal due to the charge Q is given by

$$\left(\Delta - \frac{1}{\lambda^2}\right) V(\mathbf{r}) = -\frac{Q}{\epsilon_0}\delta(\mathbf{r} - \mathbf{r}_0), \tag{3}$$

where λ is a constant called the *Debye screening length*, and $\Delta := \nabla^2$ is the Laplacian. You are given that the Green's function $G_{\hat{Y}}(\mathbf{r} - \mathbf{r}')$ of the operator $\hat{Y} := \Delta - k^2$ is

$$G_{\hat{Y}}(\mathbf{r} - \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \exp\left(-k|\mathbf{r} - \mathbf{r}'|\right). \tag{4}$$

We use the convention that a Green's function $G_{\hat{Y}}$ satisfies $\hat{Y}G_{\hat{Y}}(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$.

(d) Write the solution to Eq. (3) using the Green's function $G_{\hat{Y}}$. Give a brief physical interpretation of the solution.

Problem 2 - Radiation of a point charge (30 points)

In the Lorentz gauge, the scalar and vector potentials satisfy the equations

$$\Box V = \rho/\epsilon_0 \quad \text{and} \quad \Box \mathbf{A} = \mu_0 \mathbf{J}, \tag{5}$$

where $\Box := c^{-2}\partial_t^2 - \Delta$, is the wave operator. Recall that the retarded Green's function of the wave operator (satisfying $\Box G(\mathbf{r}-\mathbf{r}'; t-t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$) is given by

$$G(\mathbf{r}-\mathbf{r}'; t-t') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta\left(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right). \tag{6}$$

Consider a point particle with charge q moving along a trajectory $\mathbf{r}_q(t)$.

(a) What is $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ for the moving point charge?

Using the Green's function, we showed in the lectures that (you are not asked to derive this)

$$V(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R - \mathbf{R}\cdot\mathbf{v}_q/c}\right] \quad \text{and} \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi}\left[\frac{\mathbf{v}_q}{R - \mathbf{R}\cdot\mathbf{v}_q/c}\right], \tag{7}$$

where $\mathbf{R}(t) := \mathbf{r} - \mathbf{r}_q(t)$ and $\mathbf{v}_q(t) := \dot{\mathbf{r}}_q(t)$.

(b) What is the equation that determines the time at which the quantities in the square brackets of Eq. (7) are evaluated?

Why are these quantities *not* evaluated at time t?

(c) Show that

$$V(\mathbf{r}, t) \simeq \frac{q}{4\pi\epsilon_0 r}\left(1 + \hat{\mathbf{r}}\cdot\mathbf{v}_q(t - r/c)/c\right) \quad \text{and} \quad \mathbf{A}(\mathbf{r}, t) \simeq \frac{q\mu_0}{4\pi r}\mathbf{v}_q(t - r/c), \tag{8}$$

in the far-field and non-relativistic limit. State how and where these approximations are used at each step in the derivation.

Hint: You may find the following relations useful:

$$\sqrt{1+x} \simeq 1 + \frac{x}{2} \quad \text{and} \quad \frac{1}{1+x} \simeq 1 - x \quad \text{for } x \ll 1. \tag{9}$$

Suppose that you are observing the moving point charge from a large distance $r \gg r_q(t)$ and you are interested in the electromagnetic *radiation* from the particle.

(d) Explain what is meant by *the radiation fields* of the particle.

(e) It is well known that accelerated charges emit electromagnetic radiation. Can this be qualitatively seen from the expressions in Eq. (8)? (You are not required to compute $\mathbf{E}$ and $\mathbf{B}$ explicitly.)

Problem 3 - Cylindrical permanent magnet (30 points)

Consider a cylinder with radius a and length L. The cylinder has permanent and uniform magnetization M_0 parallel to its axis, and $\mathbf{J}_f = 0$ everywhere. That is,

$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & \mathbf{r} \in \text{cylinder} \\ 0 & \mathbf{r} \notin \text{cylinder} \end{cases}. \tag{10}$$

Place the coordinate system so that the cylinder extends from $-L/2 \leq z \leq L/2$, $0 \leq r \leq a$ and $0 \leq \phi < 2\pi$ (see Fig. 1).

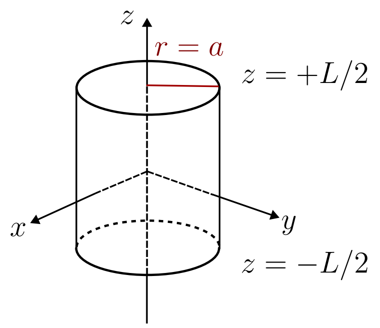

Figure 1: The figure shows the placement of the coordinate system with respect to the cylinder. The origin of the coordinate system lies at the centre of the cylinder.

(a) Argue that $\mathbf{H}$ in this case can be derived from a scalar potential $\varphi_M(\mathbf{r})$ such that $\mathbf{H}(\mathbf{r}) = -\nabla\varphi_M(\mathbf{r})$. Show that $\varphi_M(\mathbf{r})$ satisfies the Poisson equation

$$\Delta\varphi_M = -\rho_M, \tag{11}$$

where $\rho_M(\mathbf{r}) = -\nabla \cdot \mathbf{M}(\mathbf{r})$.

At this point, it might be helpful to recall that the mathematical structure of the equations is exactly the same as those of electric polarization in the absence of free charge, which can be seen by doing the replacements

$$\mathbf{M}(\mathbf{r}) \longleftrightarrow \mathbf{P}(\mathbf{r}) \quad \text{and} \quad \varphi_M(\mathbf{r}) \longleftrightarrow \epsilon_0 V(\mathbf{r}). \tag{12}$$

(b) Show that $\varphi_M(\mathbf{r})$ can be determined from the expression

$$\varphi_M(\mathbf{r}) = \frac{1}{4\pi}\int_S \mathrm{d}a' \frac{\hat{\mathbf{n}}(\mathbf{r}') \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{13}$$

where S is the surface of the cylinder, and $\hat{\mathbf{n}}$ is its outward normal.

Hint: You can either make use of the analogy in Eq. (12) directly, or note that $\mathbf{M}$ in Eq. (10) is discontinuous across the surface of the cylinder. This means that its divergence $\nabla \cdot \mathbf{M}$ is singular on the surface.

(c) Compute $\varphi_M(z)$ and use it to determine the magnetic fields $\mathbf{H}$ and $\mathbf{B}$ along the z-axis, inside and outside the cylinder. That is, compute explicit expressions for $\mathbf{H}(z)$ and $\mathbf{B}(z)$ from $\varphi_M(z)$.

NB: you should **not** find the fields as functions of the polar angle ϕ and the radius r in the xy-plane.

The z-components of the fields $H_z(z)$ and $B_z(z)$ are plotted in Fig. 2.

(d) Which of these (left or right) plots shows $B_z(z)$ and which shows $H_z(z)$? Comment on the difference between $\mathbf{H}$ and $\mathbf{B}$.

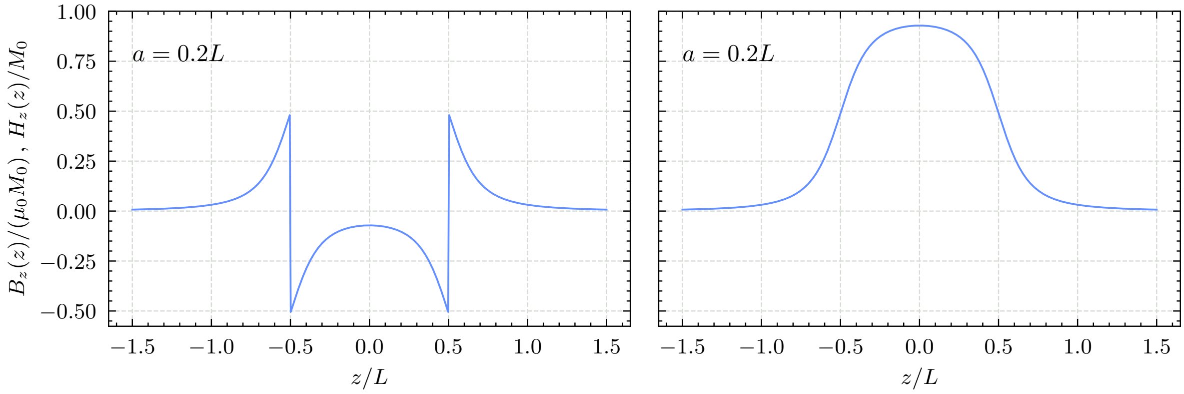

Figure 2: Plot of $H_z(z)$ and $B_z(z)$ in non-specified order with $a = 0.2L$.

Problem 4 - Gauge structure of electrodynamics (20 points)

In this course we have seen that the Maxwell theory has a *gauge structure*.

(a) What is a gauge transformation? Write down how the scalar potential $V(\mathbf{r}, t)$ and the vector potential $\mathbf{A}(\mathbf{r}, t)$ transform under such a transformation.

In class, we saw that two of the Maxwell equations in covariant form

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \tag{14}$$

followed from using the principle of least action with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}, \tag{15}$$

and the action

$$S[A] = \frac{1}{c} \int \mathrm{d}^3 r \int \mathrm{d}t \, (\mathcal{L} - A_\mu J^\mu) \, . \tag{16}$$

We are using the metric with signature $g = \mathrm{diag}(1, -1, -1, -1)$ and we have defined

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{17}$$

and

$$\partial_\mu := (\partial_t / c, \boldsymbol{\nabla}) \quad A^\mu := (V/c, \mathbf{A}) \quad \text{and} \quad J^\mu := (c\rho, \mathbf{J}). \tag{18}$$

(b) Show that $F_{\mu\nu}$ is gauge-invariant.

This implies that $\mathcal{L}$ is a gauge-invariant quantity. However, the action shown in Eq. (16) does not look gauge-invariant, because of the term where A_μ appears as itself and not via the field tensor.

(c) Derive the condition that the current J^μ has to satisfy for $S[A]$ in Eq. (16) to be gauge-invariant. Give a physical interpretation of this condition.

Hint: You can ignore boundary terms.

For some special two-dimensional materials (two spatial dimensions and one temporal dimension), the electromagnetic response is drastically different, and the Lagrangian density is given by

$$\tilde{\mathcal{L}} = \frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{19}$$

In this equation $\epsilon^{\mu\nu\rho}$ is the Levi-Civita symbol, and μ, ν, ρ are indices that take values in $\{0, 1, 2\}$. Its definition here is entirely analogous to the Levi-Civita symbol in three spatial dimensions: it is cyclic and completely antisymmetric in exchanging any pair of indices, and $\epsilon^{012} = 1$. The coefficient $k/(4\pi)$ is a real number.

In two spatial dimensions, the electric field $\mathbf{E}$ is a two-component vector defined as usual, i.e., $\mathbf{E} = -\boldsymbol{\nabla} V - \partial_t \mathbf{A}$, while the magnetic field is a scalar $B = \partial_x A_y - \partial_y A_x$. In two spatial dimensions, the electromagnetic field tensor is still given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{20}$$

and we use the metric $g = \mathrm{diag}(1, -1, -1)$.

(d) Is $\tilde{\mathcal{L}}$ in Eq. (19) invariant under gauge transformations?

Hint: It might be helpful to use that

$$\epsilon^{\mu\nu\rho} \partial_\nu A_\rho = \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho}. \tag{21}$$

(e) Show that the action

$$\tilde{S}[A] := \frac{1}{c} \int \mathrm{d}^2 r \int \mathrm{d}t \tilde{\mathcal{L}}, \tag{22}$$

is invariant under gauge transformations provided that we can ignore boundary terms.

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum: $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c} q^2 a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
$e = 1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$