NORGES TEKNISK- NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR FYSIKK

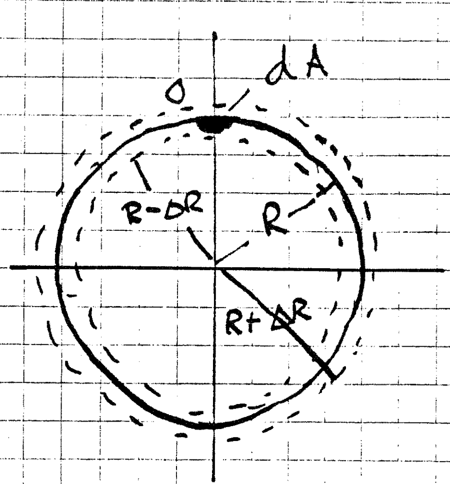
Eksamen TFY 4240: Elektromagnetisk teori

Torsdag 1 desember 2005

kl. 09.00-13.00

LØSNINGSFORSLAG

a)



Feltet i punktet O på kuleflaten er gitt som middelverdien av feltet like utenfor $(R+\Delta R)$ og like innenfor $(R-\Delta R)$ $\Delta R \to 0$. Hvorfor: se lærebok side 103.

Vi bruker Gauss sats; feltene langs de to prikkede flatene er konstant på grunn av kulesymmetrien

$$\int \vec{E}\,\vec{u}_n\,dA = E\cdot A = \frac{Q_{INNF.}}{\epsilon_0}$$

$$E(R+\Delta R) = \frac{Q}{\epsilon_0 \cdot A} \approx \frac{\sigma}{\epsilon_0}$$ når $\Delta R \to 0$

$$E(R-\Delta R) = 0$$

$\Rightarrow$ Feltet på overflaten E_0

$$E_0 = \frac{E(R+\Delta R)+E(R-\Delta R)}{2} = \frac{\sigma}{2\epsilon_0}$$

Kraften på elementet dA (Ladning $\sigma\, dA$)

$$dF = E_0\,\sigma\, dA$$

$$\underline{\underline{dF = \frac{\sigma^2}{2\epsilon_0}\,dA}}$$

b) Vi gjør kraften dF om til et trykk

$$\delta p_E = \frac{dF}{dA} = \frac{\sigma^2}{2\epsilon_0} = \frac{Q^2}{2\epsilon_0 (4\pi R^2)^2}$$

Dette settes lik trykket fra overflatespenningen

$$2\alpha/R = \left(\frac{Q}{4\pi R^2}\right)^2 \cdot \frac{1}{2\epsilon_0}$$

$$\Rightarrow \underline{\underline{Q = 8\pi R\sqrt{\alpha\epsilon_0 R}}}$$

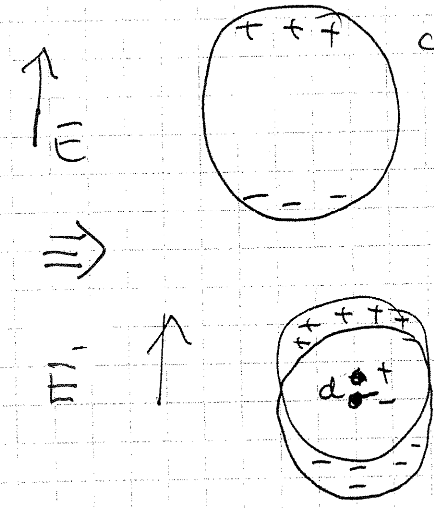
Tallverdier:

$$Q = 8\pi \cdot 5\cdot 10^{-6}\sqrt{73\cdot 10^{-3}\, 8.85\cdot 10^{-12}\cdot 5\cdot 10^{-6}}$$

$$\underline{\underline{Q = 2.25\cdot 10^{-13}\, C}}$$

c)

One sphere in an external field is equivalent to two spheres, one plus one minus charged, shifted a little; se figure

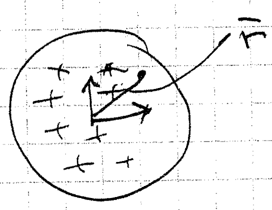


The field in the overlap region?

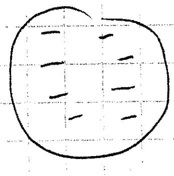
The field from one sphere, inside the sphere

Gauss

uniformly charged sphere



$$\Rightarrow \bar{E}_+ = \frac{q\bar{r}}{4\pi\epsilon_0 R^3}$$



$$\Rightarrow \bar{E}_- = \frac{-q(\bar{r}+\bar{d})}{4\pi\epsilon_0 R^3}$$

Adding the two gives a total field:

$$\overline{E}_{tot} = - \frac{q\overline{d}}{4\pi\varepsilon_0 R^3} = - \frac{\overline{p}}{4\pi\varepsilon_0 R^3}$$

The total dipole moment: $\overline{p} = q\overline{d} = \frac{4}{3}\pi R^3 \cdot \overline{P}$

$$\Rightarrow \quad \overline{E}_{tot} = - \frac{\frac{4}{3}\pi R^3 \,\overline{P}}{4\pi\varepsilon_0 R^3} = - \frac{\overline{P}}{3\varepsilon_0}$$

Outside the potential and field is the same as the field from a single dipole

$$V = \frac{1}{4\pi\varepsilon_0} \frac{\overline{p} \cdot \hat{r}}{r^2}$$

The field in the dielctric. Here we distinguish between the microscopic field and the macroscopic field. We are not interested in the details of the microscopic field, just the average ~~micr~~ macroscopic field and that is given by, as we just calculated for a sphere $\overline{E}_{inside} = -\frac{1}{3\varepsilon_0}\overline{P}$ and the total field inside

$$\overline{E} = \overline{E}_{outside} + \overline{E}_{inside}$$

Oppgave 2

a) KLASSISK MIKROSKOPISK TEORI / DRUDEMODELLEN

I et ikke-ledende isotropt medium er elektronene lokalisert, dvs. de er bundet til kjerner. I et ytre felt, e.g. et elektromagnetisk felt vil elektronene forskyves en distanse r fra sin likevektsposisjon. Dette resulterer i en *polarisasjon*

$$\overline{P} = -Ne\overline{r}$$

N er antall elektroner pr. volumenhet.

Elektronet er i følge denne klassiske Lorentz modellen bundet til atomene med en kraftkonstant k. I et metal er elektronene fri, dvs. kraftkonstanten k=0. Dette gir en bevegelsesligning

$$m\frac{d^2\overline{r}}{dt^2} + m\gamma\frac{d\overline{r}}{dt} = -e\overline{E} = -e\overline{E}_0 e^{-i\omega t}$$

γ er en dempningskonstant, m er elektronets masse. En løsning av denne differensialligningen er

$$\overline{r} = \frac{-e\overline{E}_0}{-m\omega^2 - im\omega\gamma}e^{-i\omega t}$$

og dermed blir polarisasjonen $\overline{P}$

$$\overline{P} = \frac{Ne^2/m}{-\omega^2 - i\omega\gamma}\overline{E}$$

Med $\overline{P} = \varepsilon_0\chi_e\overline{E}$ kan vi skrive

$$\overline{D} = \varepsilon_0\overline{E} + \overline{P} = \varepsilon_0(1+\chi_e)\overline{E} = \varepsilon_0\varepsilon_r\overline{E}$$

$$\Rightarrow \varepsilon_r = 1 + \chi_e$$

ε_r = dielektrisitetskonstanten

Innsatt i ligningen ovenfor gir dette:

$$\varepsilon_r = 1 + \frac{Ne^2/m\varepsilon_0}{-\omega^2 - i\omega\gamma}$$

$Ne^2/m\varepsilon_0$ har dimensjonen sek^{-2}, vi kaller den ω_p^2
Etter Drudemodellen får vi derfor:

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}$$

b) Fra potensialet $\phi = A\cos kx \cdot e^{-kz}$ fås et felt i metallet: , $\phi \equiv V$ i ex. oppg.

$$E_x^m = -\frac{\partial\phi}{\partial x} = kA\sin kx \cdot e^{-kz}$$

$$E_z^m = -\frac{\partial\phi}{\partial z} = kA\cos kx \cdot e^{-kz}$$

og i vakuum fås

$$E_x^0 = -\frac{\partial\phi_0}{\partial x} = kA\sin kx \cdot e^{kz}$$

$$E_z^0 = -\frac{\partial\phi_0}{\partial z} = -kA\cos kx \cdot e^{kz}$$

kontinuitetsbet. for $z = 0$

$E_{||}$ kontinuerlig: $E_x^v = E_x^m$

$\Rightarrow kA\sin kx = kA\sin x$ OK

D_n kontinuerlig. $D = \varepsilon E$

$$\Rightarrow \varepsilon kA\cos kx = -kA\cos kx$$

$$\Rightarrow \underline{\underline{\varepsilon = -1}}$$

For fri elektroner har vi $\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}$

$$\Rightarrow \varepsilon = -1 = 1 - \frac{\omega_p^2}{\omega^2} \Rightarrow \omega_s^2 = \frac{\omega_p^2}{2}$$

c) Fri ladningen; $z = 0$

Vi har følgende grensebet: ligning 7.63

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f$$

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f \qquad \epsilon_1 = \epsilon, \quad \epsilon_2 = 1.$$

$$\epsilon\, kA\cos kx - 1\cdot(-kA\cos kx) = \sigma_f$$

$$\sigma_f = (1+\epsilon)\, kA\cos kx \quad , \quad \epsilon = -1$$

$$\Rightarrow \sigma_f = 0$$

"Bundne" ladninger; $z = 0$, $\epsilon = -1$

Vi bruker ligning 2.36

$$\frac{\partial V_{\text{ensft}}}{\partial n} - \frac{\partial V_{\text{met}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

$$-kA\cos kx - kA\cos kx = -\frac{\sigma}{\epsilon_0}$$

$$\Rightarrow \underline{\underline{\sigma = 2kA\cos kx}}$$

Fra læreboka: $q_{ind} = -\frac{1}{2\pi}\frac{\chi_e}{\chi_e+2}\cdot q$

Ser vi at for $\chi_e = -2$ kan vi ha en indusert selv ladning for grensen $q \to 0$

$\chi_e = -2$ tilsvarer $\epsilon = 1 + \chi_e = 1 - 2 = -1$

$$\epsilon = -1$$

ↄ: overflateplasmafreq.

Opp gave 3

a) The fields are given by $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. From $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$ and $\nabla \times \nabla V \equiv 0$ the following pair of Maxwell's equations are automatically satisfied:

$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$ and $\nabla \times \mathbf{E} = -\nabla \times \nabla V - \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\frac{\partial \mathbf{B}}{\partial t}$. QED!

From the last two of Maxwell's equations we obtain:

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \Rightarrow -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \rho/\varepsilon_0 \Rightarrow$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\rho/\varepsilon_0 .$$

(A)

and: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \Rightarrow$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \mu_0 \varepsilon_0 \left(\nabla \frac{\partial V}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) \Rightarrow$$

$$\nabla^2 \mathbf{A} - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \nabla \left(\varepsilon_0 \mu_0 \frac{\partial V}{\partial t} + \nabla \cdot \mathbf{A} \right).$$

(B)

b) A gauge transform is any change of the potentials V and $\mathbf{A}$ that does *not* change the resulting fields: $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$.

Substituting the Lorentz gauge-condition, $\nabla \cdot \mathbf{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}$, into equations (A) and (B) above, we directly obtain the given wave equations:

$$\nabla^2 V - \varepsilon_0 \mu_0 \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon_0 \quad \text{and} \quad \nabla^2 \mathbf{A} - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \text{ QED!}$$

c) The solutions for V and $\mathbf{A}$ are *not* independent because the two wave equations in c) apply only when V and $\mathbf{A}$ are interrelated by the Lorentz gauge-condition. Then

ρ and **J** in the two wave equations automatically satisfy the charge conservation equation.

c) See Griffiths, example 10.2, page 425.

Example 10.2

An infinite straight wire carries the current

$$I(t) = \begin{cases} 0, & \text{for } t \leq 0, \\ I_0, & \text{for } t > 0. \end{cases}$$

That is, a constant current I_0 is turned on abruptly at $t = 0$. Find the resulting electric and magnetic fields.

Solution: The wire is presumably electrically neutral, so the scalar potential is zero. Let the wire lie along the z axis (Fig. 10.4); the retarded vector potential at point P is

$$\mathbf{A}(s, t) = \frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathscr{r}}\,dz.$$

For $t < s/c$, the "news" has not yet reached P, and the potential is zero. For $t > s/c$, only the segment

$$|z| \leq \sqrt{(ct)^2 - s^2} \tag{10.25}$$

contributes (outside this range t_r is negative, so $I(t_r) = 0$); thus

$$\begin{aligned}\mathbf{A}(s, t) &= \left(\frac{\mu_0 I_0}{4\pi}\hat{\mathbf{z}}\right)2\int_0^{\sqrt{(ct)^2-s^2}}\frac{dz}{\sqrt{s^2+z^2}} \\ &= \frac{\mu_0 I_0}{2\pi}\hat{\mathbf{z}}\ln\left(\sqrt{s^2+z^2}+z\right)\Big|_0^{\sqrt{(ct)^2-s^2}} = \frac{\mu_0 I_0}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)\hat{\mathbf{z}}.\end{aligned}$$

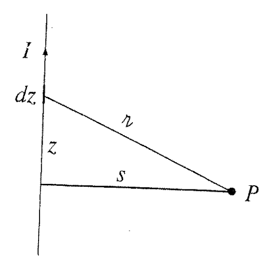


Figure 10.4

The electric field is

$$\mathbf{E}(s, t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2 - s^2}}\hat{\mathbf{z}},$$

and the magnetic field is

$$\mathbf{B}(s, t) = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\phi}} = \frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{(ct)^2 - s^2}}\hat{\boldsymbol{\phi}}.$$

Notice that as $t \to \infty$ we recover the static case: $\mathbf{E} = 0$, $\mathbf{B} = (\mu_0 I_0/2\pi s)\hat{\boldsymbol{\phi}}$.