TFY 420 ELECTROMAGNETIC THEORY
EXAM 14 DEC 2006 ; SOLUTION

PROBLEM 1

a) We first calculate the dipole moment

It is given that $\bar{p} = \int \bar{r}' \rho(r') d\tau'$
For discrete charges this takes the form $\bar{p} = \sum_i \bar{r}_i q_i = \sum_i (x_i q_i \hat{x} + y_i q_i \hat{y} + z_i q_i \hat{z})$

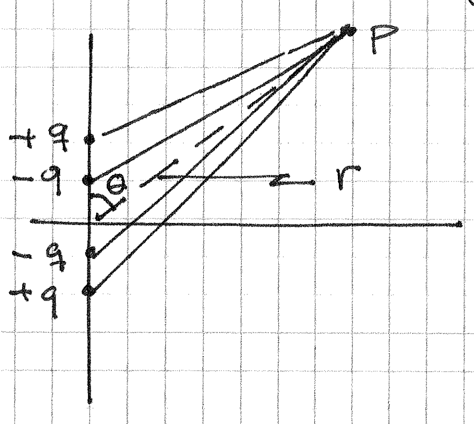
Performing this sum for the given charges gives

$$\bar{p} = \hat{x}(0) + \hat{y}(-2q \cdot a - 2q \cdot (-a)) + \hat{z}(3q \cdot a - q \cdot a)$$
$$= 2qa\,\hat{z}$$

The potential is thus:

$$V(r) = \frac{\bar{p} \cdot \hat{r}}{4\pi\varepsilon_0 r^2} = \frac{2qa\,\hat{z} \cdot \hat{r}}{4\pi\varepsilon_0 r^2} = \frac{2qa\cos\theta}{4\pi\varepsilon_0 r^2}$$

Next; calculate the potential for the four charges along the z-axis.



Here we could simply calculate $V = \sum_1^4 \frac{q_i}{4\pi\epsilon_0 r_i'}$

and then expand r_i'. I choose to use the expressions given. Thus we calculate

$$Q = \sum_1^4 q_i = +q - q - q + q = 0$$

$$\vec{p} = \sum_1^4 q_i \vec{r}_i = \left(q(-2a) - q(-a) - q \cdot a + q \cdot 2a\right)\hat{z}$$
$$= 0$$

The dipolemoment is also zero.
The lowest contribution is thus the quadrupolemoment

We calculate $Q_4 = \sum_i^4 q_i r_i'^2 \left(\frac{3}{2}\cos\theta'^2 - \frac{1}{2}\right)$

θ' is the angle between $\vec{r}'$ and $\vec{r}$ (see figure 3 in the problem set)

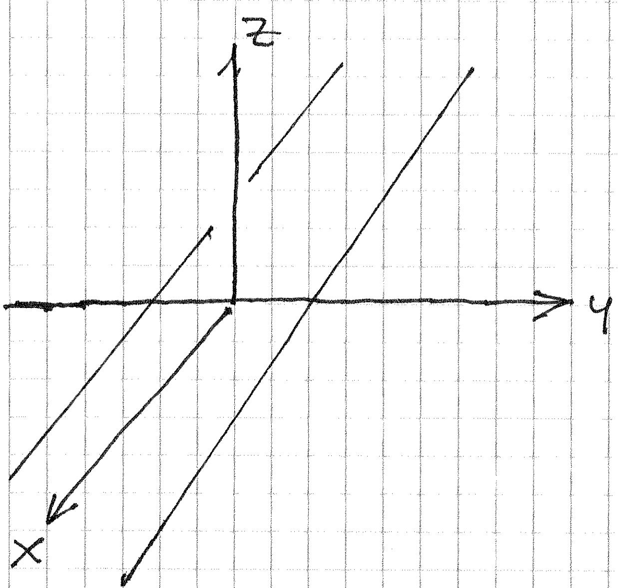
For charges with positive z, $\theta' = \theta$

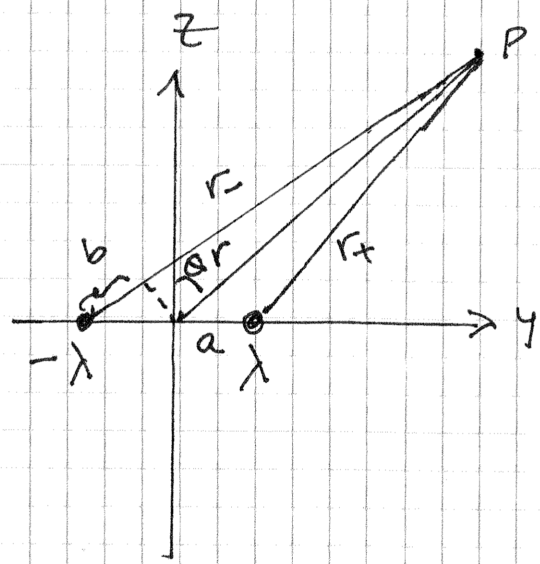
For charges with negative z, $\theta' = \pi - \theta$, but $\cos^2(\pi - \theta) = \cos^2\theta$. Thus

$$Q_4 = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \sum_1^4 r_i^2 q_i$$
$$= \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\left(q(4a^2 - a^2 - a^2 + 4a^2\right)$$
$$= (3\cos^2\theta - 1)\,3qa^2$$

and the potential is

$$V(r) = \frac{3qa^2}{4\pi\epsilon_0 r^3}(3\cos^2\theta - 1)$$





We can write

$$V(r) = -\frac{\lambda}{2\pi\epsilon_0}\ln r^+ + \frac{\lambda}{2\pi\epsilon_0}\ln r^- + c$$

We can choose $c = 0$, $b = a\sin\theta$

$$V(r) = -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_+}{r_-} = -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{r-b}{r+b}$$

$$= \frac{-\lambda}{2\pi\epsilon_0}\left(\ln\left(1-\frac{b}{r}\right) - \ln\left(1+\frac{b}{r}\right)\right)$$

We expand the logarithms

$$V(r) = \frac{-\lambda}{2\pi\epsilon_0}\left(-\frac{b}{r} - \frac{b}{r}\right)$$

$$= + \frac{\lambda a\sin\theta}{\pi\epsilon_0 r}$$

OPPGAVE 2

a) Multipliser begge sider av Faradays lov med et flateelement $d\boldsymbol{A}$ og integrer begge sider over en flate S:

$$\int_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{A} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$$

Bruk Stokes' teorem ("curl-teoremet", se vedlegg) på venstre side:

$$\int_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{A} = \oint \boldsymbol{E} \cdot d\boldsymbol{l}$$

På høyre side kan tidsderivasjonen tas utenfor integraltegnet ettersom flaten S holdes fast:

$$-\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A} = -\frac{d}{dt} \int_S \boldsymbol{B} \cdot d\boldsymbol{A}$$

Dermed er

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt} \int_S \boldsymbol{B} \cdot d\boldsymbol{A}$$

som vi skulle vise.
Multipliser begge sider av Gauss' lov for magnetfeltet med et volumelement dV og integrer begge sider over et volum:

$$\int (\nabla \cdot \boldsymbol{B})\, dV = 0$$

Bruk divergensteoremet (se vedlegg) på venstre side:

$$\int (\nabla \cdot \boldsymbol{B})\, dV = \oint \boldsymbol{B} \cdot d\boldsymbol{A}$$

Dermed er

$$\oint \boldsymbol{B} \cdot d\boldsymbol{A} = 0$$

som vi skulle vise.

b) Se f.eks. D. J. Griffiths, *Introduction to electrodynamics* (3rd ed), 7.3.6.

c) Bølgeligningen er egentlig tre ligninger:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \tilde{E}_j = \mu_0 \varepsilon_0 \frac{\partial^2 \tilde{E}_j}{\partial t^2}$$

for $j = x, y, z$. Alle tre gir oss

$$-\left(k_x^2 + k_y^2 + k_z^2\right) = -\frac{\omega^2}{c^2}$$

der $c = 1/\sqrt{\mu_0 \varepsilon_0}$ er lyshastigheten i vakuum.
Grensebetingelsen for tangentialkomponenten av $\boldsymbol{E}$ gir oss de tillatte verdiene av k_x, k_y og k_z. Inne i de metalliske veggene er $\boldsymbol{E} = 0$, så vi må ha

$$E_x(y = b) = E_x(z = d) = E_y(x = a) = E_y(z = d) = E_z(x = a) = E_z(y = b) = 0$$

(På de tre flatene $x = 0$, $y = 0$ og $z = 0$ er tangentialkomponenten av E lik null uavhengig av hva k_x, k_y og k_z er.)
Dermed må både $k_x a$, $k_y b$ og $k_z d$ være lik et helt antall ganger π, dvs

$$k_x = \frac{m\pi}{a} \quad , \quad k_y = \frac{n\pi}{b} \quad , \quad k_z = \frac{l\pi}{d}$$

med heltallige m, n og l. Bare en av disse kan være null om gangen, hvis ikke er $E = 0$, og vi har ingen elektromagnetisk bølge i det hele tatt.
Tillatte vinkelfrekvenser er med andre ord

$$\omega_{mnl} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{l^2}{d^2}}$$

d) Med $a < d < b$ er de tre laveste modene (011), (110) og (101). Innsatt tallverdier $a = 0.25$ m, $b = 0.40$ m, $d = 0.30$ m og $c = 3 \cdot 10^8$ m/s gir

$$\begin{aligned} f_{011} &= 625 \text{ MHz} \\ f_{110} &= 708 \text{ MHz} \\ f_{101} &= 781 \text{ MHz} \end{aligned}$$

Problem 3.

a) The timevariation of the source is $e^{+i\omega t}$, but at the observation point we have to replace this by $e^{i\omega t_r}$ where t_r is the retarded tim $t_r = t - r/c$

Thus $e^{i\omega t_r} = e^{i\omega(t - r/c)} = e^{i(\omega t - kr)}$

since $\omega = kc$

the conditions are those given in the textbook p 445, 446

1) $d \ll r$; The size of the dipole small compared to the distance r

2) $d \ll \lambda$; λ is the wavelength of the emitted radiation ($d \ll \frac{c}{\omega}$ in the book)

3) $\lambda \ll r$ Wavelength should be small compared to the distance from the dipole to the observation point.

b)

We have to calculate $\bar{B} = \nabla \times \bar{A}$

We use the Levi-Civita notation.

This gives

$$B_i = (\nabla \times A)_i = \delta_{ijk} \partial_j A_k$$

$$= \delta_{ijk} \partial_j \left(i\omega \frac{\mu_0}{4\pi} p_{0k} \frac{e^{i(\omega t - kr)}}{r} \right)$$

Since we are only interested in the dominating term, we only have to take the derivative of the exponential term. Thus

$$B_i = i\omega \frac{\mu_0}{4\pi} \frac{p_{0k}}{r} \partial_j e^{i(\omega t - kr)} \cdot \delta_{ijk}$$

$$= i\omega \frac{\mu_0}{4\pi} \frac{p_{0k}}{r} e^{i(\omega t - kr)} \cdot (-ik) \partial_j r \cdot \delta_{ijk}$$

$$\partial_j r = \partial_j \left(\Sigma x_i^2 \right)^{1/2} = \frac{x_j}{r}$$

$$B_i = \omega k \frac{\mu_0}{4\pi} \frac{e^{i(\omega t - kr)}}{r} \delta_{ijk} \frac{x_j p_{0k}}{r}$$

$$= \omega k \frac{\mu_0}{4\pi} \frac{e^{i(\omega t - kr)}}{r} \left[\frac{\bar{r} \times \bar{p}_0}{r} \right]_i$$

Totally we get

$$\bar{B} = \nabla \times \bar{A} = \frac{\mu_0}{4\pi} \frac{\omega^2}{c} (\hat{r} \times \bar{p}_0) \frac{e^{i(\omega t - kr)}}{r}$$

We would get the same result
in the form $-\frac{\mu_0 p_0 \omega^2}{4\pi c} \frac{e^{i(\omega t - kr)}}{r} \hat{\phi}$ if
we used the ∇ operator in spherical
coordinates given in the formula sheet

c)
$$\bar{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$

We use $B = \frac{1}{c}(\hat{r} \times \vec{E})$

$$\Rightarrow \quad \bar{S} = \frac{1}{c\mu_0}(\vec{E} \times (\hat{r} \times \vec{E}))$$
$$= \frac{1}{\mu_0 c}(\hat{r} E^2 - \vec{E}(\hat{r}\vec{E}))$$

Sine $\vec{E} \perp \hat{r}$ the last term disappear

$$\bar{S} = \frac{1}{\mu_0 c} E^2 \hat{r} = \frac{\sqrt{\epsilon_0 \mu_0}}{\mu_0} E^2 \hat{r}$$
$$= \sqrt{\frac{\epsilon_0}{\mu_0}} E^2 \hat{r}$$

The power radiated through an area $\bar{a}$ is

$$P_a = \bar{S} \cdot \bar{a}$$

The dimensions of $E^2 a^2 = (\text{Field} \cdot \text{Length})^2$
$$= \text{volt}^2$$

and the power P_a is in watts.
Comparing to Ohms law $P = \frac{V^2}{R}$

We see that $\sqrt{\frac{\mu_0}{\epsilon_0}}$ is a "resistance"

It's value is

$$R_{rad} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \cdot 10^{-7}}{8{,}85 \cdot 10^{-12}}} = 377\ \Omega$$

Looking at the expression for $\bar{B}$ we see that it can be written as

$$\bar{B} = -\frac{i\omega}{c}(\hat{r} \times \bar{A})$$

and from $\bar{E} = -c(\hat{r} \times \bar{B})$

$$\Rightarrow \bar{E} = i\omega(\hat{r} \times (\hat{r} \times \bar{A}))$$

This gives for the Poynting vector

$$\bar{S} = \frac{1}{\mu_0}(i\omega)\left(-\frac{i\omega}{c}\right)\left((\hat{r} \times (\hat{r} \times \bar{A})) \times (\hat{r} \times \bar{A})\right)$$

Call $\hat{r} \times \bar{A} = \bar{T}$

$$\bar{S} = \frac{\omega^2}{\mu_0 c}(\hat{r} \times \bar{T}) \times \bar{T} = -\frac{\omega^2}{\mu_0 c}(\bar{T} \times (\hat{r} \times \bar{T}))$$

$$= -\frac{\omega^2}{\mu_0 c}\left(\hat{r}T^2 - \bar{T}\underbrace{(\hat{r} \cdot \bar{T})}_{= 0}\right)$$

$$\bar{S} = -\frac{1}{\mu_0 c}\omega^2 \hat{r}(\hat{r} \times \bar{A})^2$$

$$= -\sqrt{\frac{\epsilon_0}{\mu_0}}\,\omega^2 (\hat{r} \times \bar{A})^2 \hat{r}$$

Thus $Y = -\sqrt{\frac{\epsilon_0}{\mu_0}}\,\omega^2$

We have used complex notation here. Then the Poynting vector should be written

$$\bar{S} = \frac{1}{\mu_0}(\bar{E} \times \bar{B}^*)$$

and the time averaged Poynting vector

$$\langle \bar{S} \rangle = \frac{1}{2\mu_0}(E \times B^*)$$

This removes the minus sign from the expression for Y, thus $(\bar{B}^* = -\bar{B})$

$$Y = \sqrt{\frac{\epsilon_0}{\mu_0}} \omega^2$$

d)

Let us first calculate the integral U

$$U = \int I_0 \frac{e^{ikz}+e^{-ikz}}{2} e^{-ikz\cos\theta} dz$$

where we have used that $\hat{r}\,\bar{r}' = z\cos\theta$ for points (source points) along the z-axis.

$$U = \frac{I_0}{2}\left[\int_{-\lambda/4}^{\lambda/4} e^{ikz(1-\cos\theta)} dz + \int_{-\lambda/4}^{\lambda/4} e^{-ikz(1+\cos\theta)} dz\right]$$

$$= \frac{I_0}{2}\left[\frac{e^{i\frac{\pi}{2}(1-\cos\theta)}-e^{-i\frac{\pi}{2}(1-\cos\theta)}}{ik(1-\cos\theta)} + \frac{e^{-i\frac{\pi}{2}(1+\cos\theta)}-e^{i\frac{\pi}{2}(1+\cos\theta)}}{-ik(1+\cos\theta)}\right]$$

$$= I_0\left[\frac{\sin\frac{\pi}{2}(1-\cos\theta)}{k(1-\cos\theta)} + \frac{\sin\frac{\pi}{2}(1+\cos\theta)}{k(1+\cos\theta)}\right]$$

$$= I_0\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{k(1-\cos\theta)} + \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{k(1+\cos\theta)}\right]$$

$$= \frac{I_0}{k}\cos\left(\frac{\pi}{2}\cos\theta\right)\frac{2}{1-\cos^2\theta}$$

$$= \frac{I_0}{k}\cos\left(\frac{\pi}{2}\cos\theta\right)\frac{2}{\sin^2\theta}$$

Thus

$$\bar{A} = \frac{\mu_0}{2\pi k} I_0 \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \cdot \frac{e^{i(\omega t - kr)}}{r}$$

This leads to a time averaged Poynting vector ($(\hat{r}\times\bar{A})^2 = \sin^2\theta\cdot A^2$)

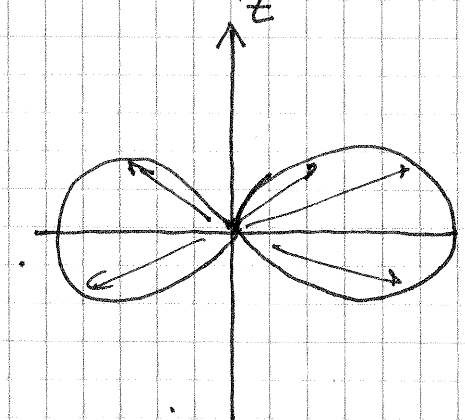
$$\langle \bar{S} \rangle = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\,\omega^2\left(\frac{\mu_0 c}{2\pi\omega}\right)^2 I_0^2\left(\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}\cdot\sin\theta\right)^2\frac{\hat{r}}{r^2}$$

$$= \frac{1}{2}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{I_0^2}{4\pi^2 r^2}\left(\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right)^2$$

and $\frac{dP}{d\Omega}$

$$\frac{dP}{d\Omega} = \sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{I_0^2}{8\pi^2}\left(\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right)^2$$

The radiation pattern.



This pattern is very similar to the short dipole pattern which goes as $\sin^2\theta$