TFY 4240 ELEKTROMAGNETISK TEORI
LØSNINGSFORSLAG, EKSAMEN 8 DES 2007

OPPGAVE 1

a) Fluksen er gitt av

$$\phi = \int_0^x B n\, dA = \int_0^x B \cdot \tfrac{1}{2}\sqrt{2} \cdot l \cdot ds$$

$$= \tfrac{1}{2}\sqrt{2}\, l \int_0^x B \cdot ds$$

$$\underline{\underline{\phi(x) = \tfrac{1}{2}\sqrt{2}\, l \; B \, x}}$$

Den induserte ems er gitt av

$$\mathcal{E} = -\frac{d\phi}{dt} = -\frac{d\phi}{dx} \cdot \frac{dx}{dt} = -\tfrac{1}{2}\sqrt{2}\, l \; B$$

$$\mathcal{E} = -\tfrac{1}{2}\sqrt{2} \cdot l \cdot B \cdot v$$

Retningen av strømmen er med-urs, og strømstyrken er:

$$\underline{\underline{I = \frac{|\mathcal{E}|}{R} = \frac{\tfrac{1}{2}\sqrt{2}\, l \cdot B \cdot v}{R}}}$$

b) Kraften er gitt av

$$\vec{F} = I \vec{l} \times \vec{B}$$

$$|\vec{F}| = I \cdot l \cdot B = \frac{\tfrac{1}{2}\sqrt{2}\, l^2 B^2 \cdot v}{R}$$

og har komponentene

$$F_y = -F_x = \frac{\tfrac{1}{2}\, l^2 B^2 \cdot v}{R}$$

Den mekaniske effekt er gitt av

$$P = |F_x| \cdot v = \frac{\frac{1}{2} l^2 B^2 \cdot v^2}{R}$$

Den elektriske effekt i motstanden R er gitt av

$$P_E = RI^2 = R \cdot \left( \frac{\frac{1}{2}\sqrt{2}\, l B\, v}{R} \right)^2$$

$$= \frac{\frac{1}{2} l^2 B^2 \cdot v^2}{R}$$

$$\Rightarrow \quad P = P_E$$

c) Vertikalkomponenten av kraften er gitt av:

$$F_y = \frac{\frac{1}{2} l^2 B^2 \cdot v}{R}$$

Staven letter fra skinnene når denne kraften er større/lik ~~[illegible]~~ tyngde~~kr~~kraften

Dette inntrer når:

$$\frac{\frac{1}{2} l^2 B^2 v}{R} \geq mg$$

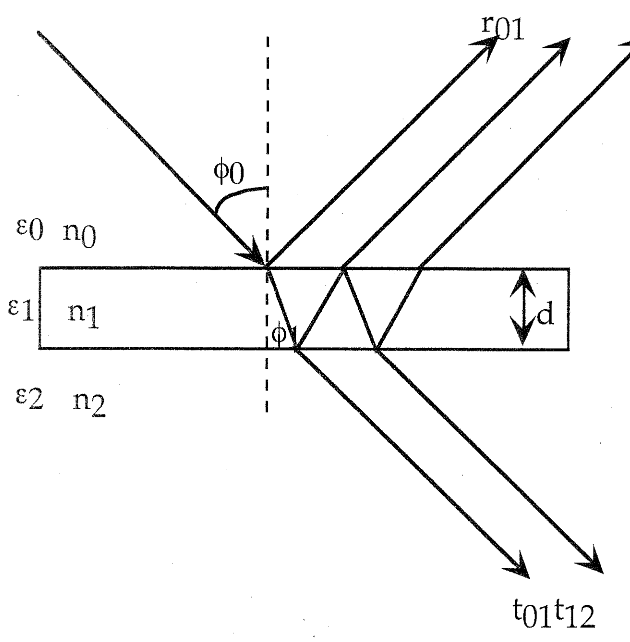
$$B^2 \geq \frac{2mgR}{l^2 v}$$

d) Anta $\bar{B} \perp \bar{I}$ og at orienteringen er slik at kraften virker vertikalt

$$\Rightarrow |F| = B \cdot I \cdot l \geq mg \Rightarrow I \geq \frac{mg}{Il} = 4905\,A$$

Oppgave 2.

Vi skal nå se litt på refleksjon og transmisjon til en tynn film på en overflate. Systemet er illustrert i figuren under.




Den innfallende bølgen $E_i$ reflekteres ved den første flaten. Noe transmitteres, reflekteres ved den indre flaten, transmitteres ved flate 1 etc. Vi får en sum av delbølger: (NB! Vi summerer feltamplituder)

$$r = r_{01} + t_{01}r_{12}t_{10}e^{2i\delta_1} + t_{01}r_{12}r_{10}r_{12}t_{10}e^{4i\delta_1} + - - - - -$$

$$= r_{01} + \frac{r_{12}t_{01}t_{10}e^{2i\delta_1}}{1 + r_{12}r_{01}e^{2i\delta_1}}$$

$$t = t_{01}t_{12}e^{i\delta_1} + t_{01}r_{12}r_{10}t_{12}e^{3i\delta_1} + - - - - -$$

$$= \frac{t_{01}t_{12}e^{i\delta_1}}{1 + r_{12}r_{01}e^{2i\delta_1}}$$

Ved å bruke Fresnels formler kan r-ligningen forenkles,

$$r = \frac{r_{01} + (r_{01}^2 r_{12} + r_{12} t_{01} t_{10})e^{2i\delta_1}}{1 + r_{12} r_{01} e^{2i\delta_1}}$$

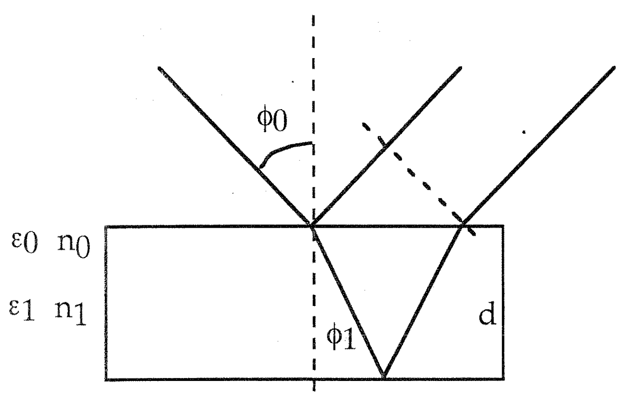
Fra Fresnels formler fås at $r_{01}^2 + t_{01}t_{10} = 1$. Dette fører da til

$$r = \frac{r_{01} + r_{12} e^{2i\delta_1}}{1 + r_{12} r_{01} e^{2i\delta_1}} \qquad \text{Vis dette!}$$

$2\delta_1$ er faseforskjellen mellom bølge 1 og 2. Ut fra geometri/optisk veilengdeforskjell blir den

$$\delta_1 = \frac{2\pi}{\lambda} n_1 d_1 \cos\varphi_1$$

Bevis:




Forskjellen i optisk veilengde er gitt av figuren:

$$\frac{n_1 2d}{\cos\phi_1} - 2d\,\mathrm{tg}\,\phi_1 \sin\phi_0 n_0$$

$$= 2d\left(\frac{n_1}{\cos\phi_1} - \mathrm{tg}\,\phi_1 \sin\phi_0 n_0\right)$$

$$= 2d\left(\frac{n_1}{\cos\phi_1} - \frac{\sin\phi_1}{\cos\phi_1} n_1 \sin\phi_1\right) = 2dn_1 \cos\phi_1$$

Dette gir da $\delta_1 = \frac{2\pi}{\lambda} n_1 d_1 \cos\varphi_1$ som angitt ovenfor.

b) We assume normal incidence
The reflection coefficients are then

$$r_{01} = \frac{n_0 - n_1}{n_0 + n_2} \qquad r_{12} = \frac{n_1 - n_2}{n_1 + n_2}$$

Insert this into

$$\frac{r_{01} + r_{12}}{1 + r_{01} r_{12}} = \frac{\left(\frac{n_0 - n_1}{n_0 + n_1}\right) + \left(\frac{n_1 - n_2}{n_1 + n_2}\right)}{1 + \left(\frac{n_0 - n_1}{n_0 + n_1}\right)\left(\frac{n_1 - n_2}{n_1 + n_2}\right)}$$

$$= \frac{2n_1(n_0 - n_2)}{2n_1(n_0 + n_2)} = \frac{n_0 - n_2}{n_0 + n_2} = r_{02}$$

This must also be so from the equation for a thin film on a surface

$$r = \frac{r_{01} + r_{12} e^{2i\delta_1}}{1 + r_{01} r_{12} e^{2i\delta_1}}$$

If the film thickness is zero $\delta_1 = 0$

$$r = \frac{r_{01} + r_{12}}{1 + r_{01} r_{12}}$$

and this corresponds to the reflection at the 0-2 interface.

c)

Dersom $2\delta_1$ er lik $2\pi$ "ser" ikke strålingen boksen

$$\delta_1 = \frac{2\pi}{\lambda} n_1 d_1 \cos\varphi_1 \qquad \varphi_1 = 0$$

$$\Rightarrow \quad 2\pi = 2 \cdot \frac{2\pi}{\lambda} n_1 d_1$$

$$d_1 = \frac{\lambda}{2 \cdot n_1} = \frac{c}{2\nu n_1} = \frac{3 \cdot 10^8}{2 \cdot 10^{10} \cdot 2{,}5} = 0{,}6 \cdot 10^{-2}\,m$$

$$= 6\,mm$$

Problem 3.

a) The timevariation of the source is $e^{+i\omega t}$, but at the observation point we have to replace this by $e^{i\omega t_r}$ where $t_r$ is the retarded tim $t_r = t - r/c$

Thus $e^{i\omega t_r} = e^{i\omega(t - r/c)} = e^{i(\omega t - kr)}$

since $\omega = kc$

the conditions are those given in the textbook p 445, 446

1) $d \ll r$ ; The size of the dipole small compared to the distance $r$

2) $d \ll \lambda$ ; $\lambda$ is the wavelength of the emitted radiation ($d \ll \frac{c}{\omega}$ in the book)

3) $\lambda \ll r$ Wavelength should be small compared to the distance from the dipole to the observation point.

b)

The vector potential is now given by

$$\bar{A}(\bar{r},t) = \frac{\mu_0}{4\pi} \frac{e^{i(\omega t - kr)}}{r} \int \bar{J}(\bar{r}') e^{ik\bar{r}'\hat{r}} d\tau'$$

Let the antenna be located along the z-axis. We have to evaluate the integral

$$I = \int \bar{J}(r') e^{ikz'\cos\theta} dz'$$

$$= I_0 \int_0^{\lambda/2} \frac{e^{ikz'} - e^{-ikz'}}{2} e^{ikz'\cos\theta} dz'$$

$$+ I_0 \int_{-\lambda/2}^{0} \frac{e^{ikz'} - e^{-ikz'}}{2} e^{ikz'\cos\theta} dz'$$

These integrals are given:

$$I = I_0 \left( \frac{1 + e^{i\pi\cos\theta}}{k\sin^2\theta} + \frac{1 + e^{-i\pi\cos\theta}}{k\sin^2\theta} \right)$$

which leads to a vector potential $\bar{A}$

$$\bar{A}(r,t) = \frac{\mu_0}{4\pi} \frac{e^{i(\omega t - kr)}}{r} \frac{I_0 \cdot 2}{k\sin^2\theta} \left(1 + \cos(\pi\cos\theta)\right) \hat{z}$$

c)

$$\bar{S} = \sqrt{\frac{\epsilon_0}{\mu_0}}\, \omega^2 (\hat{r} \times \bar{A})^2\, \hat{r}$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}}\, \omega^2 \sin^2\theta \left(\frac{\mu_0}{2\pi}\frac{c}{\omega}\right)^2 \frac{(1+\cos(\pi\cos\theta))^2}{\sin^4\theta} \frac{\hat{r}}{r^2} I_0^2$$

and the time averaged Poynting vector becomes

$$\langle \bar{S} \rangle = \frac{1}{2}\sqrt{\frac{\mu_0}{\epsilon_0}} \frac{I_0^2}{4\pi^2 r^2} \frac{(1+\cos(\pi\cos\theta))^2}{\sin^2\theta}\, \hat{r}$$

and

$$\frac{dP}{d\Omega} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{I_0^2}{8\pi^2} \frac{(1+\cos(\pi\cos\theta))^2}{\sin^2\theta}$$

The total radiated power becomes

$$P = \int \frac{dP}{d\Omega} d\Omega = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{I_0^2}{8\pi^2} \int \frac{(1+\cos(\pi\cos\theta))^2}{\sin^2\theta} \sin\theta\, d\theta\, d\varphi$$

$$= \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{I_0^2}{4\pi}\, 1{,}66$$

The radiation is max for $\theta = \pi/2$.

$$S_{max} = \frac{1}{2}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{I_0^2}{4\pi^2 r^2}\cdot 4$$

and the directivity

$$D = \frac{4\pi r^2 S_{max}}{P} = \frac{\frac{1}{2}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{I_0^2}{4\pi^2 r^2}\cdot 4 \quad 4\pi r^2}{\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{I_0^2}{4\pi}\cdot 1{,}66} = 4{,}82$$

The radiation resistance:

$$P = \frac{1}{2} I_0^2 R_{rad}$$

$$\Rightarrow R_{rad} = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{I_0^2}{4\pi}\cdot 1{,}66\cdot 2/I_0^2$$

$$= \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{3{,}32}{4\pi} = 377\,\frac{3{,}32}{4\pi} = \underline{99{,}6\,\Omega}$$

d)

i)

Vi ser direkte fra uttrykket for $\bar{A}$ at $\bar{A} = 0$

$\bar{r}' \times \bar{J}(r') = 0$ siden $\bar{r}'$ under integreringen ligger langs z-aksen og $\bar{J} \parallel \hat{z}$-aksen.

$$M_0 = 0$$

ii)

Now remember that the magnetisation due to a current $\bar{J}$ was $\bar{M} = \frac{1}{2}(\bar{r}' \times \bar{J})$

and the total moment: $\bar{m} = \frac{1}{2}\int(\bar{r}' \times \bar{J})\, d\tau'$ (prob. 5.60)

The magnetic moment of a current loop was $\bar{m}_0 = \pi a^2 \cdot I_0 \hat{z}$ and the expression above is the magnetic moment/volume of a distributed current

Example: Current loop. Use cylindrical coordinates

$$\bar{J}(\bar{r}') = I_0\, \delta(r' - a)\,\delta(z' - z_0)\,\hat{\varphi}$$

$$\bar{m} = \frac{1}{2}\int(\bar{r}' \times \hat{\varphi})\, I_0\, \delta(r' - a)\,\delta(z' - z_0)\; r'dr'd\varphi\, dz'$$

$$= \frac{1}{2}\, 2\pi I_0 a^2\, \hat{z} = I_0 \pi a^2\, \hat{z} =$$ moment of current loop