TFY4240 : Exam Dec 2013

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Problem 1

a) The current density is given by

$$\vec{J}(\vec{r}) = \frac{\hat{z} I}{\pi(b^2-a^2)} \Theta(r_{\parallel}-a)\Theta(b-r_{\parallel}) \quad (1\text{-}1)$$

b) The problem is a magnetostatic prob. By symmetry it follows that

$$\vec{H} \propto \hat{\phi}$$

and the magnitude of the H-field follows from Amperes law by applying it to a circular Amperian loop of radius $R_{\parallel}$ and placed in the xy-plane

$$\nabla \times \vec{H} = \vec{J}$$

$$\int d\vec{A} \cdot \nabla \times \vec{H} = \int d\vec{A} \, \vec{J}$$

$$\oint d\vec{\ell} \cdot \vec{H} = 2\pi R_{\parallel} H = \int d\vec{A} \cdot \vec{J}$$

$$\Rightarrow H(R_{\parallel}) = \frac{1}{2\pi R_{\parallel}} \int_0^{R_{\parallel}} dr_{\parallel} \, r_{\parallel} \int_0^{2\pi} d\phi \; J(r_{\parallel})$$

$$= \frac{1}{R_{\parallel}} \frac{I}{\pi(b^2-a^2)} \int_a^{R_{\parallel}} dr_{\parallel} \, r_{\parallel} \;\; \Theta(b-r_{\parallel})$$

i) $R_{\parallel} < a$: $\quad H(R_{\parallel}) = 0 \qquad$ (since $\vec{J} = 0$)

ii) $a < R_{\parallel} < b$

$$H(R_{\parallel}) = \frac{1}{R_{\parallel}} \frac{I}{\pi(b^2-a^2)} \frac{1}{2}(R_{\parallel}^2 - a^2)$$

$$= \frac{I}{2\pi R_{\parallel}} \frac{R_{\parallel}^2 - a^2}{b^2-a^2}$$

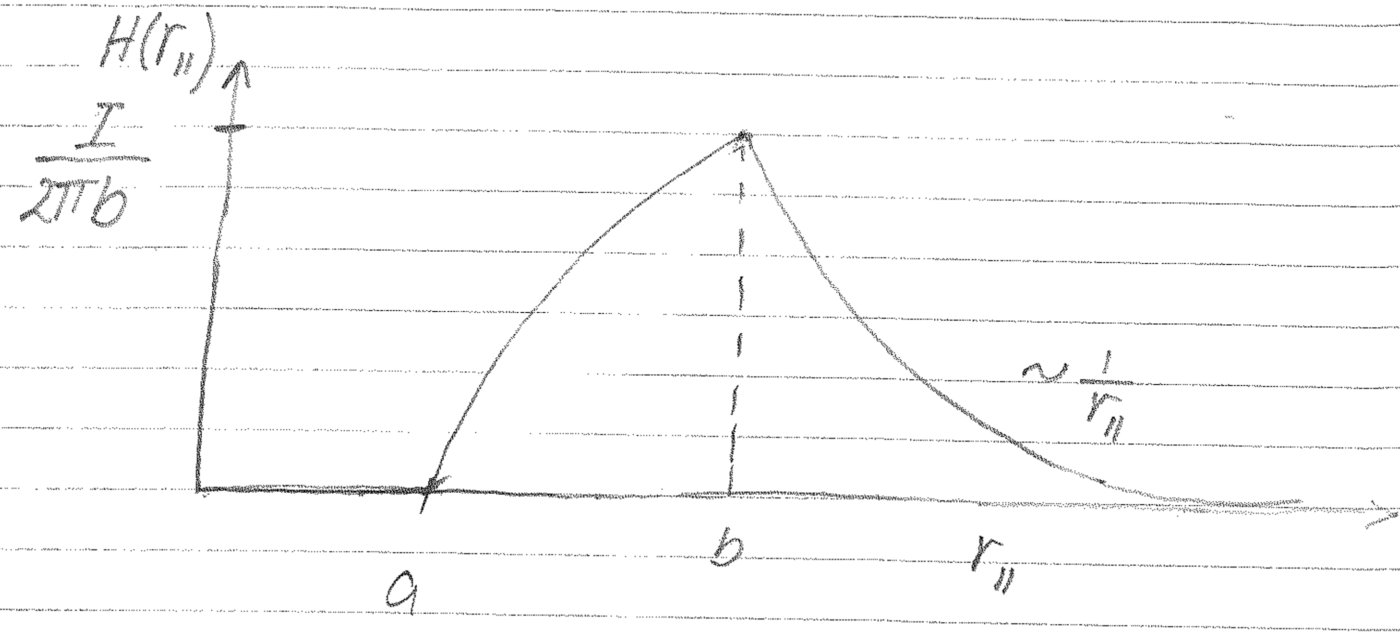
iii) $R_{\parallel} > b$:

$$H(R_{\parallel}) = \frac{I}{2\pi R_{\parallel}}$$

Renaming $R_{\parallel} \to r_{\parallel}$ it follows

$$H(\vec{r}_{\parallel}) = \hat{\phi} \frac{I}{2\pi r_{\parallel}} \begin{cases} 0 & r_{\parallel} < a \\ \dfrac{r_{\parallel}^2 - a^2}{b^2-a^2} & a < r_{\parallel} < b \\ 1 & r_{\parallel} > b \end{cases}$$

Sketch



c) Since this is a magnetostatic problem ; $\vec{E}(\vec{r}) = 0$ everywhere. This follows since the source of the electric field, $\rho(\vec{r})$, is zero everywhere.

Problem 2

a) We need to solve Laplace eq. which is a seperable PDE. Since the problem has azimuthal symmetry the general solution of LE is given by Eq. (1)

Terms of the form r^ℓ and $1/r^{\ell+1}$ appear in this eq. since they are the independent solutions of the radial eq. $P_\ell(\cos\theta)$ denotes the Legendre polynomials.

b) The boundary conditions on $V(\vec{r})$ are

$$(i)\ r=0 : \quad V(\vec{r}) \text{ is finite} \qquad (2\text{-}1)$$

$$(ii)\ r\to\infty : \quad V(\vec{r}) \text{ corresponds to } \vec{E}_0 = -\nabla V$$

$$\Rightarrow V(\vec{r}) = -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta) \qquad (2\text{-}2)$$

$$(iii)\ r=a : \quad V^{(1)}(\vec{r})\Big|_{r=a} = V^{(2)}(\vec{r})\Big|_{r=a} \qquad (2\text{-}3)$$

$$\varepsilon_1\left[\partial_r V^{(1)}(\vec{r})\right]\Big|_{r=a} = \varepsilon_2\left[\partial_r V^{(2)}(\vec{r})\right]\Big|_{r=a} \qquad (2\text{-}4)$$

This means that $V(\vec{r})$ and $\varepsilon\partial_r V(\vec{r})$ are continuous over the surface.

Here $V^{(n)}(\vec{r})$ denotes the potential in region n.

c) We start by writing the potential in the two regions in the general form

$$V^{(1)}(\vec{r}) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta) \qquad (2\text{-}5)$$

and

$$V^{(2)}(\vec{r}) = \sum_{\ell=0}^{\infty} \left[C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta) \qquad (2\text{-}6)$$

Now from BC (2-1) it follows $B_\ell = 0$ for all ℓ's. Moreover, from (2-2) we find

$$C_\ell = \begin{cases} -E_0 & \ell = 1 \\ 0 & \ell > 1 \end{cases} \qquad (2\text{-}7)$$

From the BC on the surface of the sphere one gets:

$$\sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta)$$

$$= -E_0 a P_1(\cos\theta) + \sum_{\ell=0}^{\infty} \frac{D_\ell}{a^{\ell+1}} P_\ell(\cos\theta) \qquad (2\text{-}8)$$

and

$$\varepsilon_1 \sum_{\ell=0}^{\infty} \ell A_\ell a^{\ell-1} P_\ell(\cos\theta) \qquad (2\text{-}9)$$

$$= -\varepsilon_2 E_0 P_1(\cos\theta) - \varepsilon_2 \sum_{\ell=0}^{\infty} (\ell+1) \frac{D_\ell}{a^{\ell+2}} P_\ell(\cos\theta)$$

which should be satisfied for any value of θ.

Now introducing $x = \cos\theta$ and operating from the left with the operator

$$\int_{-1}^{1} dx \; P_m(x)$$

on these relations gives after using the orthogonality condition

$$\int_{-1}^{1} dx \; P_m(x) P_n(x) = \frac{2}{2m+1} \delta_{mn}$$

$$A_m a^m = -E_0 a \, \delta_{m1} + D_m a^{-(m+1)}$$

$$\varepsilon_1 m A_m a^{m-1} = -\varepsilon_2 E_0 \delta_{m1} - \varepsilon_2 (m+1) D_m a^{-(m+2)}$$

or

$$\begin{bmatrix} a^m & -a^{-(m+1)} \\ \varepsilon_1 m a^{m-1} & \varepsilon_2 (m+1) a^{-(m+2)} \end{bmatrix} \begin{bmatrix} A_m \\ D_m \end{bmatrix}$$

$$= -\delta_{m1} E_0 \begin{bmatrix} a \\ \varepsilon_2 \end{bmatrix} \qquad (2\text{-}10)$$

When $m \neq 0$ it follows immediately that

$$A_m = D_m = 0 \qquad m \neq 1$$

However, for $m=1$:

$$\begin{bmatrix} a & -a^{-2} \\ \varepsilon_1 & 2\varepsilon_2 a^{-3} \end{bmatrix} \begin{bmatrix} A_1 \\ D_1 \end{bmatrix} = -E_0 \begin{bmatrix} a \\ \varepsilon_2 \end{bmatrix} \tag{2-11}$$

a system that has the solution

$$\begin{bmatrix} A_1 \\ D_1 \end{bmatrix} = \frac{-a^2 E_0}{\varepsilon_1 + 2\varepsilon_2} \begin{bmatrix} 2\varepsilon_2 a^{-3} & -\varepsilon_1 \\ a^{-2} & a \end{bmatrix}^T \begin{bmatrix} a \\ \varepsilon_2 \end{bmatrix}$$

$$= \frac{-a^2 E_0}{\varepsilon_1 + 2\varepsilon_2} \begin{bmatrix} +2\varepsilon_2 a^{-2} + \varepsilon_2 a^{-2} \\ -\varepsilon_1 a + \varepsilon_2 a \end{bmatrix}$$

$$= \frac{E_0}{\varepsilon_1 + 2\varepsilon_2} \begin{bmatrix} -3\varepsilon_2 \\ +(\varepsilon_1 - \varepsilon_2) a^3 \end{bmatrix} \tag{2-12}$$

Therefore, the potential becomes :

$$V(\vec{r}) = \begin{cases} -\dfrac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} E_0 r \cos\theta , & r < a \\ -\left[1 - \dfrac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \left(\dfrac{a}{r}\right)^3\right] E_0 r \cos\theta & r > a \end{cases}$$

(2-13)

$$K_1 = \frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \qquad k_2 = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}$$

Alternatively, one could have derived this expression by using that the expressions in front of $P_\ell(\cos\theta)$ in Eqs. (2-8) and (2-9) have to be the same on both sides of the eqs.

d) The electric field is given by

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

Inside the sphere one finds the constant field

$$\vec{E}(\vec{r}) = \frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\vec{E}_0 \quad , \quad r < a \qquad (2\text{-}14)$$

To get the field outside the sphere, we start by calculating $\nabla\left(\frac{\vec{E}_0\cdot\vec{r}}{r^3}\right)$:

$$\partial_i\left(\frac{\vec{E}_0\cdot\vec{r}}{r^3}\right) = E_{0,k}\,\frac{(\partial_i r_k)\,r^3 - \partial_i(r^3)\,r_k}{r^6}$$

$$= E_{0,k}\,\frac{\delta_{ik}\,r^3 - 3r^2\,\partial_i(r)\,r_k}{r^6}$$

$$\partial_i(r) = \partial_i\sqrt{x_m x_m} = \frac{2x_m\delta_{im}}{2r} = \frac{x_i}{r}$$

$$= E_{0,k}\,\frac{\delta_{ik}\,r^3 - 3r^2\,[\hat{r}]_i\,r_k}{r^6}$$

$$= \frac{E_{0,i} - 3(\vec{E}_0\cdot\hat{r})\,[\hat{r}]_i}{r^3} \qquad (2\text{-}15)$$

Thus

$$\nabla\left(\frac{\vec{E_o}\cdot\vec{r}}{r^3}\right) = \frac{\vec{E_o} - 3(\vec{E_o}\cdot\hat{r})\hat{r}}{r^3}$$

so that the field outside the sphere becomes $(r > a)$

$$\vec{E}(\vec{r}) = \vec{E_o} + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} a^3 \left[\frac{3(\vec{E_o}\cdot\hat{r})\hat{r} - \vec{E_o}}{r^3}\right] \tag{2-16}$$

e] The interpretation of Eq (2-16) is that the first term is the external field while the 2nd term result from the polarization of the sphere that $\vec{E_o}$ causes inside the sphere. Actually a dipole moment is induced in the sphere.

The field from an infinitesimal dipole of dipole moment $\vec{p}$ is given by*

$$\frac{1}{4\pi\varepsilon_0\varepsilon_2}\,\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3}$$

Hence we conclude that the induced dipole moment in the sphere is

$$\vec{p} = 4\pi\varepsilon_0 a^3\, \varepsilon_2\, \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\, \vec{E_o} \tag{2-17}$$

We note that $\vec{p} \propto \vec{E_o}$

* Here ε_0 is the permittivity of free space

The same conclusion can be reached by noting that the 2nd term of the potential $V(\vec{r})$ for $r > a$ has the form of an electric dipole potential

$$\frac{1}{4\pi\varepsilon_0\varepsilon_2} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

The polarizability becomes:

$$\alpha = 4\pi\varepsilon_0 a^3 \varepsilon_2 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \qquad (2\text{-}18)$$

The polarization, $\vec{P}$, of the sphere is defined as dipole moment per unit volum Therefore, we have that

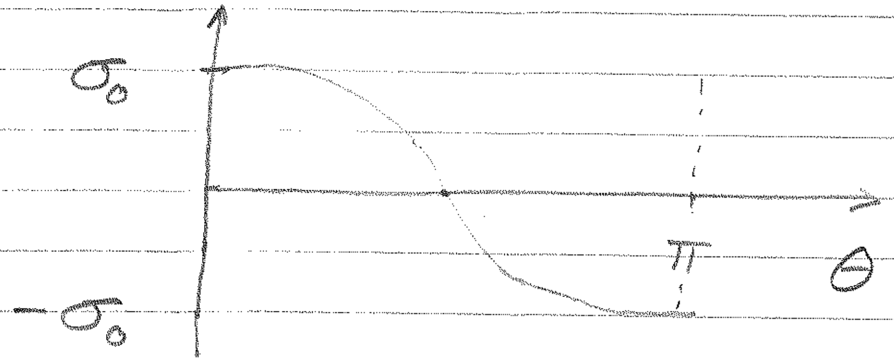
$$\vec{P} = \frac{\vec{p}}{\frac{4}{3}\pi a^3} = 3\varepsilon_0 \varepsilon_2 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \vec{E_0} \qquad (2\text{-}19)$$

f] The charge density induced at the surface of the sphere is given by

$$\sigma(\theta) = \hat{r}\cdot\vec{P}$$

$$= 3\varepsilon_0 \varepsilon_2 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} E_0 \cos\theta$$

$\sigma(\theta) = \sigma_0 \cos\theta$



The positive/negative charge is induced in the same/oposite direction of the field. Maximum/minimum is at $\theta = 0 / \theta = \pi$ and $\sigma(\theta)$ is zero for $\theta = \pi/2$. The solution has axial sym. This behavior is as expected.

g) Since $\sigma(\theta)$ has axial symmetry, i.e., it does not depend on the azimuthal angle ϕ, a rotation of the sphere about E_0 will not give rise to a (polarization) curret.

Hence, no magnetic field will be produced due to the rotation.

Problem 3

a) In general one has (in the Lorentz gauge)

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}'|\omega)\, e^{-i\omega t_r}}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\mu_0}{4\pi} e^{-i\omega t} \int d^3r' \, \frac{\vec{J}(\vec{r}'|\omega)\, e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \qquad (3\text{-}1)$$

In the far field ($kr \gg 1$, $r \gg r'$) we approximate

$$\frac{1}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{r} \qquad (3\text{-}2)$$

$$k|\vec{r}-\vec{r}'| = k\sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}$$

$$= kr\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{\hat{r}\cdot\vec{r}'}{r}}$$

$$\simeq kr\left(1 - \frac{\hat{r}\cdot\vec{r}'}{r}\right) \qquad (3\text{-}3)$$

Eqs (3-2) and (3-3) subsituted into (3-1) gives the result that should be demonstrated.

$$\vec{A}(\vec{r},t) \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr-i\omega t}}{r} \vec{f}(\theta,\phi)$$

where

$$\vec{f}(\theta,\phi) = \int d^3r' \, \vec{J}(\vec{r}'|\omega) \, e^{-ik\hat{r}\cdot\vec{r}'}.$$

b) It follows directly from the geometry that

$$\vec{J}(\vec{r},t) = J(\vec{r},t) \cdot \hat{\phi}$$

Since the radius of the loop is $r=a$ and it is placed in the xy-plane it follows that in sperical coordinates

$$J(\vec{r},t) \propto \delta(r-a)\,\delta(\theta-\pi/2)$$

Moreover, $J(\vec{r},t)$ must be prop. to $I(t)$ so we write,

$$J(\vec{r},t) = \alpha \, I(t)\,\delta(r-a)\,\delta(\theta-\pi/2)$$

where α is a constant to be determined. We note that $[\alpha] = m^{-1}$ in order for J to have the correct dimension.

So, how can we determine α? We know that

$$\int d\vec{A} \, \vec{J}(\vec{r},t) = I(t)$$

By integrating over, for instance, the x^+z-plane we get (using polar coordinates)

$$\int_0^\infty dr\, r \int_0^\pi d\theta\ \alpha\, I(t)\, \delta(r-a)\delta(\theta-\pi/2) = \alpha a I(t)$$

Since this integral should be equal to $I(t)$ it follows that $\alpha = 1/a$. Hence, we conclude

$$\vec{J}(\vec{r},t) = \frac{I(t)}{a}\delta(r-a)\delta(\theta-\pi/2)\,\hat{\phi} \qquad (3\text{-}4)$$

[The same result can be obtain by e.g. integrating $\vec{J}(\vec{r},t)$ over all of space since this integral should be $\propto I(t)2\pi a$]

c) From Eq. (9b) it follows $[\vec{m}(t) = \vec{m}_0 e^{-i\omega t}]$

$$\vec{m}_0 = \frac{1}{2}\int d^3r\, \vec{r}\times\vec{J}(\vec{r}|\omega)$$

$$= \frac{\hat{z}}{2}\frac{I_0}{a}\int r\,\delta(r-a)\,\delta(\theta-\pi/2)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{\hat{z}}{2}\frac{I_0}{a}\, 2\pi a^3$$

$$= \hat{z}\, I_0\, \pi a^2 \qquad (3\text{-}5)$$

so that

$$\vec{m}(t) = \hat{z}\, I_0\, \pi a^2 e^{-i\omega t} \qquad (3\text{-}6)$$

which is the time-dependent dipole moment.

d) With this expression for $\vec{m}_0$ we get for

$$\vec{f}(\theta,\phi) = -ik\,\vec{m}_0 \times \hat{r}$$

$$= -ik\, m_0 \sin\theta$$

$$= -ik\, I_0 \pi a^2 \sin\theta \qquad (3\text{-}7)$$

so that the vector potential becomes in the far field regions

$$\vec{A}(\vec{r};t) = \frac{\mu_0}{4\pi} \frac{e^{ikr-i\omega t}}{r} \left(-ik I_0 \pi a^2 \sin\theta\right)\hat{\phi}$$

$$= -\frac{ik\mu_0 I_0 a^2}{4} \sin\theta \frac{e^{ikr-i\omega t}}{r}\,\hat{\phi} \qquad (3\text{-}8)$$

which has the behavior we should show.

d) Radiation fields are electromagnetic (E- or H-fields) which decay like $1/r$ in the far field region.

Since the fields can be calculated from $\vec{A}(\vec{r},t)$, and we only here are interested in radiation fields, it is sufficient to use $\lim_{kr\to\infty} \vec{A}(\vec{r},t)$ for this calculation.

The magnetic field is given by

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r},t)$$

Using Eq (3-8) one gets in the far field

$$\vec{H}(\vec{r},t) = \frac{-ikI_0a^2}{4} \nabla\times\left(\sin\theta \frac{e^{ikr-i\omega t}}{r}\vec{\phi}\right)$$

$$= \frac{-ikI_0a^2}{4}\left[-\partial_r\left\{\sin\theta \frac{e^{ikr-i\omega t}}{r}\right\}\hat{\theta}\right.$$

$$\left. + O(1/r^2)\right]$$

$$= \frac{ikI_0a^2}{4}\sin\theta\,(ik\cdot r - 1)\frac{e^{ikr-i\omega t}}{r^2}\hat{\theta}$$

$$+ O(1/r^2)$$

$$= -\frac{k^2I_0a^2}{4}\sin\theta\frac{e^{ikr-i\omega t}}{r}\hat{\theta} + O(1/r^2)$$

(3-9)

Here the radiation H-field is the first term without the $O(1/r^2)$. Note from (3-9) that this radiation field also can be written as:

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t) = \frac{1}{\mu_0} i\vec{k}\times\vec{A}(\vec{r},t)$$

with $\vec{k} = k\hat{r}$ and using $\hat{r}\times\hat{\phi} = -\hat{\theta}$.

The latter result follows from observing that $\vec{A}(\vec{r},t)$ has the form of a spherical wave with an angular dependent amplitude that locally in the far field behaves like a plane wave.

The electric field (in the far field) follows from Eq. (3-9) by the use of ampers law:

$$-i\omega\varepsilon_0 \vec{E}(\vec{r},t) = \nabla \times \vec{H}$$

$$\vec{E}(\vec{r},t) = \frac{i}{\omega\varepsilon_0} \nabla \times \vec{H}$$

$$= \frac{i}{\omega\varepsilon_0} i\vec{k} \times \vec{H}$$

$$= \frac{-1}{\omega\varepsilon_0} \vec{k} \times \vec{H} \quad , \vec{k} = k\hat{r}$$

$$= \frac{k^3 I_0 a^2}{4\omega\varepsilon_0} \sin\theta \, \frac{e^{ikr-i\omega t}}{r} \underbrace{\hat{r}\times\hat{\theta}}_{\hat{\phi}} \qquad (3\text{-}10)$$

This result one could also have obtained by the direct method used to calculate $\vec{H}$.

e) The time average Poynting vector (in the far field) becomes

$$\langle S \rangle = \frac{1}{2} \vec{E} \times \vec{H}^*$$

$$= +\frac{1}{2}\frac{1}{\omega\varepsilon_0} \vec{H}^* \times (\vec{k} \times \vec{H})$$

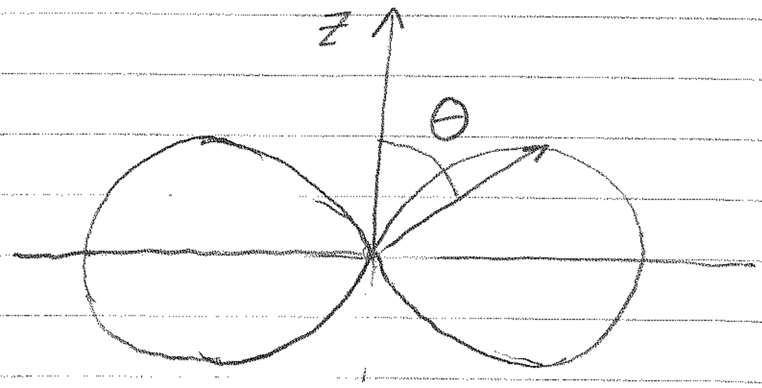
$$= \frac{1}{2}\frac{1}{\omega\varepsilon_0} \left[\vec{k} |\vec{H}|^2 - \vec{H} \underbrace{(\vec{H}^* \cdot \vec{k})}_{0} \right]$$

$$= \frac{1}{2}\frac{k}{\omega\varepsilon_0} \frac{k^4 I_0^2 a^4}{16} \frac{\sin^2\theta}{r^2} \hat{r}$$

$$= \frac{\mu_0 \omega^4 I_0^2 a^4}{32 c^3} \frac{\sin^2\theta}{r^2} \hat{r} \qquad (3\text{-}11)$$

$$\frac{dP}{d\Omega} = r^2\,\hat{r}\cdot\langle \vec{S}\rangle$$

$$= \frac{\mu_0 \omega^4 I_0^2 a^4}{32\,c^3}\sin^2\theta \qquad (3\text{-}12)$$



f] The total power radiated by the system is :

$$P = \int d\Omega\,\frac{dP}{d\Omega}$$

$$= \frac{\mu_0 \omega^4 I_0^2 a^4}{32\,c^3}\int_0^{\pi} d\theta\,\sin\theta \int_0^{2\pi} d\phi\,\sin^2\theta$$

$$= \frac{\mu_0 \omega^4 I_0^2 a^4}{32\,c^3}\,2\pi \int_0^{\pi} \sin^3\theta\,d\theta$$

$$= \frac{\mu_0 \omega^4 I_0^2 a^4}{32\,c^3}\,2\pi\,\underbrace{\int_{-1}^{1}(1-x^2)\,dx}_{4/3}\,,\quad x=\cos\theta$$

$$= \frac{\pi \mu_0 \omega^4 I_0^2 a^4}{12\,c^3} \qquad (3\text{-}13)$$

g) Since $P \propto I_0^2 a^4$ it follows if $I_0 \rightarrow \alpha I_0$ then $a \rightarrow a/\sqrt{\alpha}$ for the product $I_0^2 a^4$ to remain unchanged.

Therefore if I_0 is doubled, the radius has to be reduced by a factor $\sqrt{2}$ in order for the radiated power to remain unchanged.

Since $P \propto m_0^2$, the amplitude of the dipole moment, it follows that if m_0 is unchanged, so is P. Moreover, since $m_0 = I_0 \cdot A$, where $A = \pi a^2$, our conclusion made above is reasonable.