

TFY4240

Exam 2014 Dec 3.

Problem 1

a) The charge density is:

$$\rho(\vec{r}) = q \delta(\vec{r} - \hat{x}_3 d)$$

The eq. to be satisfied are

* above the surface:

$$\nabla \cdot \vec{D} = \rho(\vec{r}); \quad \nabla \cdot \vec{E} = \rho(\vec{r})/\epsilon_0 \epsilon_r, \\ \nabla \times \vec{E} = 0$$

* below the surface

$$\nabla \cdot \vec{D} = 0; \quad \nabla \cdot \vec{E} = 0 \\ \nabla \times \vec{E} = 0$$

b) Since $\vec{E} = -\nabla V(\vec{r})$ under electrostatic conditions, it follows that

$$\nabla^2 V(\vec{r}) = \begin{cases} -\frac{\rho(\vec{r})}{\epsilon_0 \epsilon_r} & x_3 > 0 \\ 0 & x_3 < 0 \end{cases}$$

The eq. $\nabla \times \vec{E}$ is always satisfied.

Boundary conditions

i) $x_3 = 0$

$$V_+(\vec{r})|_{\vec{r}=\vec{r}_0} = V_-(\vec{r})|_{\vec{r}=\vec{r}_0}$$

$$\epsilon_1 \partial_3 V_+(\vec{r})|_{\vec{r}=\vec{r}_0} = \epsilon_2 \partial_3 V_-(\vec{r})|_{\vec{r}=\vec{r}_0}$$

ii) $r \rightarrow \pm \infty$

$$V(\vec{r}) \rightarrow 0$$

c) Image method:

Since the metal is grounded we choose:

$$V_-(\vec{r}) = 0$$

and the method of images gives for the potential in region 1:

$$V_+(\vec{r}) = \frac{1}{4\pi\epsilon_0\epsilon_1} \left[\frac{q}{|\vec{r} - d\hat{x}_3|} - \frac{q}{|\vec{r} + d\hat{x}_3|} \right]$$

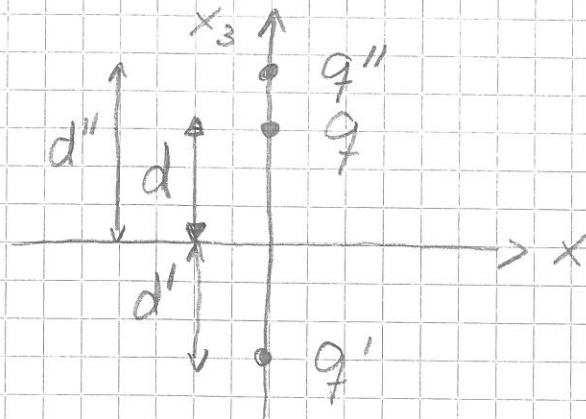
where

$$|\vec{r} \pm d\hat{x}_3| = \sqrt{x_0^2 + (x_3 \pm d)^2}$$

One may check explicitly that the BC are satisfied at $x_3 = 0$

d) In this dielectric case the potential at $x_3=0$ is no longer zero. Hence the situation is somewhat more complicated:

For symmetry reasons any image charges are expected to be placed on the x_3 -axis.



$$V_+(r) = \frac{1}{4\pi\epsilon_0\epsilon_1} \left[\frac{q}{|\vec{r} - d\hat{x}_3|} + \frac{q'}{|\vec{r} + d'\hat{x}_3|} \right]$$

where

$$|\vec{r} - d\hat{x}_3| = \sqrt{x_{''}^2 + (x_3 - d)^2}$$

$$x_{''}^2 = x_1^2 + x_2^2$$

$$|\vec{r} + d'\hat{x}_3| = \sqrt{x_{''}^2 + (x_3 + d')^2}$$

$$V_-(r) = \frac{1}{4\pi\epsilon_0\epsilon_2} \frac{q''}{|\vec{r} - d''\hat{x}_3|}$$

$$|\vec{r} - d''\hat{x}_3| = \sqrt{x_{''}^2 + (x_3 - d'')^2}$$

e) The BC at infinity is satisfied. However, we have to impose the BC at $x_3=0$

$$\begin{aligned} \text{ii)} \quad & \frac{1}{\epsilon_1} \left[\frac{q}{\sqrt{x_{11}^2 + d^2}} + \frac{q'}{\sqrt{x_{11}^2 + d'^2}} \right] \\ &= \frac{1}{\epsilon_2} \frac{q''}{\sqrt{x_{11}^2 + d''^2}} \end{aligned} \quad (1.1a)$$

$$\begin{aligned} \text{iii)} \quad & \partial_3 [x_{11}^2 + (x_3 - a)^2]^{-1/2} \\ &= -\frac{1}{2} [x_{11}^2 + (x_3 - a)^2]^{-3/2} 2(x_3 - a) \\ &= \frac{x_3 - a}{[x_{11}^2 + (x_3 - a)^2]^{3/2}} \end{aligned}$$

$$\begin{aligned} & \frac{q(-d)}{[x_{11}^2 + d^2]^{3/2}} + \frac{q'd'}{[x_{11}^2 + d'^2]^{3/2}} \\ &= \frac{q''(-d'')}{[x_{11}^2 + d''^2]^{3/2}} \end{aligned} \quad (1.1b)$$

Eq (1.1) should hold for any x_{11} , in part. for $x_{11}=0$ and $x_{11} \gg d, d'$ and d'' (or $x_{11} \rightarrow \infty$)

$$\frac{q}{\epsilon_1 d} + \frac{q'}{\epsilon_2 d'} = \frac{q''}{\epsilon_2 d''} \quad (1.2a)$$

$$-\frac{qd}{d^3} + \frac{q'd'}{d'^3} = -\frac{q''d''}{d''^3}$$

$$-\frac{q}{d^2} + \frac{q'}{d'^2} = -\frac{q''}{d''^2} \quad (1.2b)$$

Equation (1.2) represents two eqs. for four unknown. Hence we have an under-determined system.

We use this to try to find a solution when

$$d = d' = d''$$

This choice is motivated by what one has in the metallic case.

Now, Eq. (1.2) becomes

$$\frac{q}{\epsilon_1} + \frac{q'}{\epsilon_2} = \frac{q''}{\epsilon_2} \quad (1.3a)$$

$$-q + q' = -q'' \quad (1.3b)$$

or

$$\begin{bmatrix} \frac{1}{\epsilon_1} & -\frac{1}{\epsilon_2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q' \\ q'' \end{bmatrix} = \begin{bmatrix} -\frac{1}{\epsilon_1} \\ 1 \end{bmatrix} q$$

The solution is:

$$q' = - \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} q$$

$$q'' = \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1} q$$

and the potential in the two regions is determined.

[1] If $|\epsilon_2| \gg \epsilon_1$, one finds

$$q' = - \frac{1 - \frac{\epsilon_1}{\epsilon_2}}{1 + \frac{\epsilon_1}{\epsilon_2}} q \approx -q$$

$$q'' = \frac{2}{1 + \frac{\epsilon_1}{\epsilon_2}} q \approx 2q$$

Hence, the potential $V_-(\vec{r})$ will become small and $V_+(\vec{r})$ approaches what we have for the potential above a grounded metal plate.

In the extreme case $|\epsilon_2| \rightarrow \infty$ V_- becomes zero.

Problem 2

a) Maxwell's eqs. with sources

$$\nabla \cdot \vec{D} = \rho \quad (2.1a)$$

$$\nabla \cdot \vec{B} = 0 \quad ("b)$$

$$\nabla \times \vec{E} = -\partial_t \vec{B} \quad ("c)$$

$$\nabla \times \vec{H} = \vec{J} + \partial_t \vec{D} \quad ("d)$$

(2.1d) $\times \vec{B}$ - $\vec{D} \times$ (2.1c) gives:

$$(\nabla \times \vec{H}) \times \vec{B} - \vec{D} (\nabla \times \vec{E}) = \vec{J} \times \vec{B} + (\partial_t \vec{D}) \times \vec{B} + \vec{D} \times \partial_t \vec{B}$$

$$\begin{aligned} & \partial_t (\vec{D} \times \vec{B}) + \vec{J} \times \vec{B} \\ & + \vec{D} \times (\nabla \times \vec{E}) + \vec{B} \times (\nabla \times \vec{H}) - \underbrace{[\vec{E}(\nabla \cdot \vec{D}) - \rho \vec{E}]}_0 \\ & - \vec{H}(\nabla \cdot \vec{B}) = 0 \end{aligned}$$

$$\begin{aligned} & \partial_t (\vec{D} \times \vec{B}) + \rho \vec{E} + \vec{J} \times \vec{B} \\ & + \{ \vec{D} \times (\nabla \times \vec{E}) - \vec{E}(\nabla \cdot \vec{D}) + \vec{B} \times (\nabla \times \vec{H}) - \vec{H}(\nabla \cdot \vec{B}) \} = 0 \end{aligned}$$

Hence, Eq (1a) of the exam set is derived and we find

$$\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B}$$

The vector $\vec{f}$ describes the force density acting at position $\vec{r}$ at time t .

b] Lets look at the i 'th component

$$[\vec{D} \times (\nabla \times \vec{E}) - \vec{E}(\nabla \cdot \vec{D})]_i$$

$$= \epsilon_{ijk} D_j (\nabla \times \vec{E})_k - E_i (\nabla \cdot \vec{D})$$

$$= \epsilon_{ijk} D_j \epsilon_{kmn} \partial_m E_n - E_i (\nabla \cdot \vec{D})$$

$$= \epsilon_{kij} \epsilon_{kmn} D_j \partial_m E_n - E_i (\nabla \cdot \vec{D})$$

$$= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) D_j \partial_m E_n - E_i (\nabla \cdot \vec{D})$$

$$= \underbrace{D_j \partial_i E_j - D_j \partial_j E_i}_{\frac{1}{2} \partial_i (\vec{E} \cdot \vec{D})} - \underbrace{E_i \partial_j D_j}_{\partial_j (E_i D_j)}$$

$$= -\partial_j (E_i D_j) + \frac{1}{2} \partial_i (\vec{E} \cdot \vec{D})$$

$$= \partial_j \left[E_i D_j + \delta_{ij} \frac{\vec{E} \cdot \vec{D}}{2} \right]$$

Similar expression for the magnetic term
 $(\vec{E} \rightarrow \vec{H}, \vec{D} \rightarrow \vec{B})$

Therefore

$$\{ \}_{j} = \partial_i T_{ij}$$

with

$$T_{ij} = D_i E_j + B_i H_j + \delta_{ij} \left(\frac{\vec{D} \cdot \vec{E}}{2} + \frac{\vec{B} \cdot \vec{H}}{2} \right)$$

Since $\vec{D} = \epsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$ this tensor is indeed symmetric

c) Integrating Eq (2a) of the exam set over a small volume V gives

$$\int_V d^3r (\partial_t \vec{g} + \vec{f}) + \int_V d^3r \nabla \cdot \vec{T} = 0$$

$$\frac{d}{dt} \vec{G} + \vec{F} + \int_{\partial V} d\vec{S} \cdot \vec{T} = 0$$

This means that the time rate of change of the total momentum inside V plus the total force acting on the volume

ADD

Problem 3

3-7

a) Dipole moment

$$\begin{aligned}\vec{p}(t) &= \int d^3r \vec{r} \rho(\vec{r}, t) \\ &= e^{-i\omega t} \int d^3r \vec{r} Q [\delta(x_3 - a) - \delta(x_3 + a)] \delta(x_1) \delta(x_2) \\ &= e^{-i\omega t} Q \hat{x}_3 (a - (-a)) \\ &= 2aQ e^{-i\omega t} \hat{x}_3\end{aligned}\quad (3.1)$$

The current density can be obtained from the continuity eq.

$$\nabla \cdot \vec{J} = -\partial_t \rho(\vec{r}, t) = -i\omega \rho(\vec{r}, t)$$

Since $\vec{J} \propto \hat{x}_3$ one gets by a simple integration

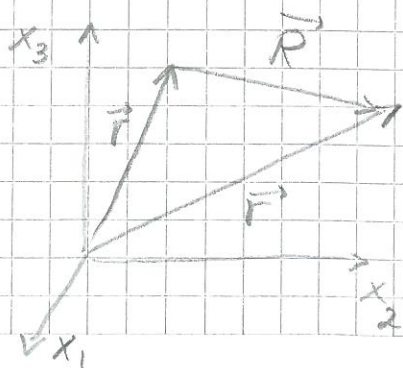
$$\begin{aligned}\vec{J}(\vec{r}|\omega) &= -i\omega \int dx_3 \rho(\vec{r}|\omega) \hat{x}_3 \\ &= -i\omega Q \delta(x_1) \delta(x_2) \Theta(a - |x_3|) \hat{x}_3\end{aligned}\quad (3.2)$$

b) $\vec{r}$: observation point

$\vec{r}'$: source point

R : source-observer separation: $\vec{R} = \vec{r} - \vec{r}'$

t_r : retarded time: $t_r = t - R/c$



The radiation zone is the region where $kR \gg 1$ (and $ka \ll 1$). In this region radiation fields will dominate if present.

$$\begin{aligned} R &= |\vec{r} - \vec{r}'| \\ &= r \left(1 - 2 \frac{\hat{r} \cdot \vec{r}'}{r} + \frac{r'^2}{r^2} \right)^{1/2} \\ &\approx r - \hat{r} \cdot \vec{r}' \end{aligned}$$

$$\frac{1}{R} \approx \frac{1}{r} \left(1 + \frac{\hat{r} \cdot \vec{r}'}{r} + \dots \right)$$

In the radiation zone the vector potential becomes:

$$\begin{aligned} \vec{A}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}', \omega)}{R} e^{-i\omega(t - \frac{R}{c})} \\ &\approx \frac{\mu_0}{4\pi} \frac{e^{-i\omega t}}{r} \int d^3r' \vec{J}(\vec{r}', \omega) e^{ikr} e^{-ik\hat{r} \cdot \vec{r}'} \\ &= \frac{\mu_0}{4\pi} \frac{e^{ikr - i\omega t}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int d^3r' \vec{J}(\vec{r}', \omega) (\hat{r} \cdot \vec{r}')^n \end{aligned}$$

c) The lowest order contribution is:

$$\begin{aligned}
 \vec{A}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \frac{e^{+ikr-i\omega t}}{r} \int d^3r' \vec{J}(\vec{r}') / \omega \\
 &= -i\omega Q \frac{\mu_0}{4\pi} \frac{\exp(ikr-i\omega t)}{r} \int_{-a}^a dx_3' \hat{x}_3 \\
 &= -i\omega \frac{\mu_0}{4\pi} 2a Q \frac{\exp(ikr-i\omega t)}{r} \hat{x}_3 \\
 &= \frac{\mu_0}{4\pi} \dot{\vec{p}}(t) \frac{e^{ikr}}{r}
 \end{aligned}$$

The magnetic field (in the radiation zone)

$$\begin{aligned}
 \vec{H}(\vec{r}, t) &= \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r}, t) \\
 H_i &= \frac{1}{\mu_0} \epsilon_{ijk} \partial_j \left(\frac{\mu_0}{4\pi} \dot{\vec{p}}_k \frac{e^{ikr}}{r} \right) \\
 &= \frac{1}{4\pi} \epsilon_{ijk} \dot{\vec{p}}_k \partial_j \left(\frac{\exp(ikr)}{r} \right) \\
 &= \frac{1}{4\pi} \epsilon_{ijk} \dot{\vec{p}}_k \frac{ik}{r} \partial_j r + O(1/r^2) \\
 \partial_j r &= \partial_j \sqrt{x_i x_i} = (\hat{r})_j \\
 &= \frac{1}{4\pi} \epsilon_{ijk} \hat{r}_j \dot{\vec{p}}_k \frac{e^{ikr}}{r} (ik) \\
 &= \frac{ik}{4\pi} (\hat{r} \times \dot{\vec{p}})_i \frac{e^{ikr}}{r} \\
 &= \frac{ck^2}{4\pi} (\hat{r} \times \vec{p}(t))_i \frac{e^{ikr}}{r}
 \end{aligned}$$

Hence in the radiation zone one gets

$$\vec{H}(\vec{r}, t) = \frac{ck^2}{4\pi} (\hat{r} \times \vec{p}(t)) \frac{e^{ikr}}{r}$$

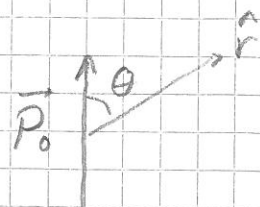
The electric field becomes
(from Ampere's law)

3-4

$$\begin{aligned}\vec{E}(\vec{r}, t) &= \frac{1}{-i\omega\epsilon_0} \nabla \times \vec{H} \\ &= \frac{1}{-i\omega\epsilon_0} i\vec{k} \times \vec{H} + O(1/r^2) \\ &= \frac{1}{\omega\epsilon_0} \frac{ck^2}{4\pi} k \hat{r} \times (\hat{r} \times \vec{p}(t)) \frac{e^{ikr}}{r} \\ &= \frac{k^2}{4\pi\epsilon_0} \hat{r} \times [\hat{r} \times \vec{p}(t)] \frac{e^{ikr}}{r}\end{aligned}$$

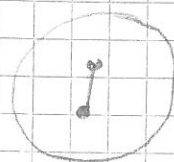
The time averaged Poynting's vector:

$$\begin{aligned}\langle \vec{S} \rangle &= \frac{1}{2} \vec{E} \times \vec{H}^* \\ &= \frac{1}{2} \frac{1}{\omega\epsilon_0} (\vec{k} \times \vec{H}) \times \vec{H}^* \\ &= \frac{1}{2} \frac{1}{\omega\epsilon_0} \vec{H}^* \times (\vec{k} \times \vec{H}) \\ &= \frac{1}{2} \frac{1}{\omega\epsilon_0} [\vec{k} |\vec{H}|^2 - \vec{H} (\underbrace{\vec{k} \cdot \vec{H}^*}_0)] \\ &= \frac{1}{2} \frac{k}{\omega\epsilon_0} \frac{c^2 k^4}{(4\pi)^2} \frac{|\hat{r} \times \vec{p}_0|^2}{r^2} \hat{r}, \quad \vec{p}_0 = 2aQ \hat{x}_3 \\ &\quad \vec{k} = k \hat{r} \\ &= \frac{1}{32\pi^2\epsilon_0} \frac{\omega^4}{c^3} \frac{|\hat{r} \times \vec{p}_0|^2}{r^2} \hat{r} \\ &= \frac{1}{8\pi^2\epsilon_0} \frac{\omega^4}{c^3} a^2 Q^2 \frac{\sin^2\theta}{r^2} \hat{r}\end{aligned}$$



d) The time-averaged power crossing a surface A is:

$$\begin{aligned} P &= \int \langle \vec{S} \rangle d\vec{A} \\ &= \int \langle \vec{S} \rangle \cdot \hat{r} r^2 d\Omega \\ &\equiv \int \left\langle \frac{dP}{d\Omega} \right\rangle d\Omega \end{aligned}$$



$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \langle \vec{S} \rangle \cdot \hat{r} r^2$$

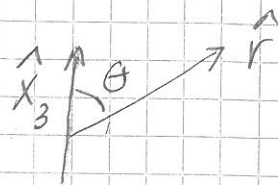
$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{8\pi^2 \epsilon_0} \frac{\omega^4 q^2 Q^2}{c^3} \sin^2 \theta$$

e) In the Lorentz gauge one has

$$\begin{aligned} \vec{A}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}', t_r)}{R} \quad R = |\vec{r} - \vec{r}'| \\ &= \frac{\mu_0}{4\pi} (-i\omega Q) \int d^3r' \delta(x'_1) \delta(x'_2) \Theta(a - |x'_3|) \hat{x}_3 \\ &\quad e^{-i\omega(t - |\vec{r} - \vec{r}'|/c)} / |\vec{r} - \vec{r}'| \\ &= -i \frac{\mu_0 \omega Q}{4\pi} e^{-i\omega t} \int_{-a}^a dx'_3 \frac{e^{ik|\vec{r} - \vec{x}'_3 \hat{x}_3|}}{|\vec{r} - \vec{x}'_3 \hat{x}_3|} \hat{x}_3 \end{aligned}$$

f] Under the assumption $r \gg a$ it follows that
(since $|x_3'| < a$)

$$\begin{aligned} |\vec{r} - x_3' \hat{x}_3| &= [r^2 - 2\vec{r} \cdot \hat{x}_3 x_3' + x_3'^2]^{1/2} \\ &= r \left[1 - 2 \frac{x_3'}{r} \cos\theta + \left(\frac{x_3'}{r}\right)^2 \right]^{1/2} \\ &\approx r \left(1 - \frac{x_3'}{r} \cos\theta + \dots \right) \\ &= r - x_3' \cos\theta + \dots \end{aligned}$$



However in the expression

$$\frac{1}{|\vec{r} - x_3' \hat{x}_3|} \approx \frac{1}{r}$$

This gives :

$$\begin{aligned} \vec{A}(\vec{r}, t) &= -\frac{i\mu_0 \omega Q}{4\pi} e^{-i\omega t} \\ &\quad \times \int_{-a}^a dx_3' \frac{e^{ikr - ikx_3' \cos\theta}}{r} \hat{x}_3 \\ &= -\frac{i\mu_0 \omega Q}{4\pi} e^{ikr - i\omega t} \underbrace{\int_{-a}^a dx_3' e^{-ikx_3' \cos\theta}}_{\hat{x}_3} \\ &= \frac{2}{k \cos\theta} \sin(ka \cos\theta) \end{aligned}$$

$$= -i \frac{\mu_0 \omega Q}{4\pi} \frac{2}{k} \frac{e^{ikr-i\omega t}}{r} \frac{\sin(ka \cos\theta)}{\cos\theta} \hat{x}_3$$

$$= -i \frac{\mu_0 c Q}{2\pi} \frac{e^{ikr-i\omega t}}{r} \frac{\sin(ka \cos\theta)}{\cos\theta} \hat{x}_3$$

g) In spherical coordinates

$$\hat{x}_3 = \cos\theta \hat{r} - \sin\theta \hat{\theta}$$

and since $\vec{A}$ is independent of ϕ it follows that:

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A}$$

$$\approx \frac{1}{\mu_0} i\vec{k} \times \vec{A} \quad (\text{in the radiation zone})$$

The electric field follows from Ampere's law.

$$\nabla \times \vec{H} = -i\omega \vec{D} = -i\omega \epsilon_0 \vec{E}$$

$$\Rightarrow \vec{E} = \frac{1}{-i\omega \epsilon_0} \nabla \times \vec{H}$$

$$\approx \frac{1}{-i\omega \epsilon_0} i\vec{k} \times \vec{H} \quad (\text{radiation})$$

$$= -\sqrt{\frac{\mu_0}{\epsilon_0}} \hat{k} \times \vec{H}$$

$$\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^*$$

$$= \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{H}^* \times (\hat{k} \times \vec{H})$$

$$= \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \left[\hat{k} |\vec{H}|^2 - \vec{H} (\underbrace{\vec{H}^* \cdot \hat{k}}_0) \right]$$

$$= \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} |\vec{H}|^2 \hat{k}$$

$$= \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{k^2}{\mu_0^2} |\hat{k} \times \vec{A}|^2 \hat{r}, \quad \hat{k} = \hat{r}$$

$$= \frac{1}{2} \frac{\omega^2}{c^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{\mu_0^2} |\vec{A}|^2 \sin^2 \theta \hat{r}$$

$$= \frac{\omega^2 Q^2}{8\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{r^2} \frac{\sin^2(ka \cos \theta)}{\cos^2 \theta} \sin^2 \theta \hat{r}$$

$$\underline{h)} \quad \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^2 Q^2}{8\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\sin^2(ka \cos \theta)}{\cos^2 \theta} \sin^2 \theta$$

if When $ka \ll 1$ it follows that

$$\sin(ka \cos \theta) \approx ka \cos \theta$$

so that

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^2 Q^2}{8\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} k^2 a^2 \sin^2 \theta$$

Rewriting the prefactor

$$\omega^2 Q^2 \sqrt{\frac{\mu_0}{\epsilon_0}} k^2 a^2$$

$$= \omega^2 Q^2 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\omega^2}{c^2} a^2$$

$$= \frac{\omega^4 Q^2 a^2}{c^3}$$

gives

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^4 a^2 Q^2}{8\pi^2 \epsilon_0 c^3} \sin^2 \theta$$

This is the same expression we obtained in 3d]