

Solution for exam TFY4240, June 2017

Problem 1

$$(a) \quad \nabla \cdot \vec{D} = \rho_f \quad \text{and} \quad \vec{D} = \epsilon \vec{E}$$

$$\Rightarrow \quad \nabla \cdot (\epsilon \vec{E}) = \rho_f$$

Inside a medium m , $\epsilon = \epsilon_m$ (a constant)

$$\Rightarrow \quad \epsilon_m \nabla \cdot \vec{E} = \rho_f$$

In electrostatics, $\vec{E} = -\nabla V$

$$\Rightarrow \quad \underline{\underline{\nabla^2 V = -\rho_f / \epsilon_m}}$$

$$(b) \quad \vec{E} = -\nabla V \Rightarrow dV = -\vec{E} \cdot d\vec{r}$$

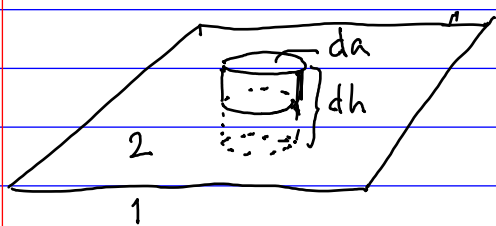
Integrating this from a point $\vec{r}_1$ in medium 1 to a point $\vec{r}_2$ in medium 2 gives

$$V_2 - V_1 = \int_1^2 dV = - \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r} \quad (\text{path-indep.})$$

Letting $\vec{r}_1$ and $\vec{r}_2$ approach the same point on the boundary (i.e. $\vec{r}_2 - \vec{r}_1 \rightarrow 0$), the integral $\rightarrow 0$
Since $\vec{E}$ is finite

$$\Rightarrow \quad \underline{\underline{V_2 - V_1 = 0}}$$

Next, we introduce a Gaussian "pillbox"-shaped surface which straddles the boundary, as shown in the figure below:



Gauss's law on integral form:

$$\oint_A \vec{D} \cdot d\vec{a} = Q_{f, \text{inside}}$$

Considering the limit of a "wafer-thin" box (height $dh \rightarrow 0$)

(1) The only contribution to the surface integral comes from the top & bottom parts (parallel to boundary), each of area da .

$$\Rightarrow \oint \vec{D} \cdot d\vec{a} = (\vec{D}_2 - \vec{D}_1) \cdot \hat{n} da = (\epsilon_2 \vec{E}_2 - \epsilon_1 \vec{E}_1) \cdot \hat{n} da$$

(2) The only contribution to $Q_{f, \text{inside}}$ can come from free charge at the boundary, with surface charge density σ_f :

$$\Rightarrow Q_{f, \text{inside}} = \sigma_f da$$

Cancelling da gives $(\epsilon_2 \vec{E}_2 - \epsilon_1 \vec{E}_1) \cdot \hat{n} = \sigma_f$

Using $\vec{E} = -\nabla V$ and $\partial_n V = \hat{n} \cdot \nabla V$ then gives

$$\underline{\underline{\epsilon_2 \partial_n V_2 - \epsilon_1 \partial_n V_1 = -\sigma_f}}$$

(c) As $r \rightarrow \infty$, $\vec{E} = -\nabla V \rightarrow \vec{E}_0 = E_0 \hat{z}$

$\Rightarrow V \rightarrow -E_0 z + C$ as $r \rightarrow \infty$

We take $C=0$ so that in the absence of $\vec{E}_0$, this boundary condition for V reduces to $V \rightarrow 0$.
As $z = r \cos \theta$ we thus get

$V(\vec{r}) \rightarrow -E_0 r \cos \theta = -E_0 r P_1(\cos \theta)$ as $r \rightarrow \infty$

Comparing this with the expansion gives ($o = \text{outside}$)

$A_1^{(o)} = -E_0$ and $A_l^{(o)} = 0$ for $l \neq 1$

$\Rightarrow V_{(o)}(\vec{r}) = -E_0 r \cos \theta + \sum_l \frac{B_l^{(o)}}{r^{l+1}} P_l(\cos \theta)$

In the expansion of $V_{(i)}(\vec{r})$ ($i = \text{inside}$), $B_l^{(i)}$ must be 0 for all l to prevent $V_i(\vec{r})$ from diverging as $r \rightarrow 0$. Thus

$V_{(i)}(\vec{r}) = \sum_{l=0}^{\infty} A_l^{(i)} r^l P_l(\cos \theta)$

In the following, rename $A_l^{(i)} \equiv A_l$ and $B_l^{(o)} \equiv B_l$.
To find these we must use Eqs. (2) and (3) in the text (with $m=1$ inside, $m=2$ outside)
First Eq. (2): $V_{(o)} = V_{(i)}$ for $r=R$

$\Rightarrow -E_0 R \cos \theta + \sum_l \frac{B_l}{R^{l+1}} P_l(\cos \theta) = \sum_l A_l R^l P_l(\cos \theta)$

Equating coefficients of P_l for each l gives

$l=1: -E_0 R + \frac{B_1}{R^2} = A_1 R \Rightarrow \underline{B_1 = R^3 (A_1 + E_0)}$ (*)

$l \neq 1: \frac{B_l}{R^{l+1}} = A_l R^l \Rightarrow \underline{B_l = R^{2l+1} A_l}$ (**)

Next, for Eq. (3), note that $\partial_n = \partial_r$. We need

$$\partial_r V_{(o)} = -E_0 \cos \theta + \sum_l (-1)(l+1) \frac{B_l}{r^{l+2}} P_l(\cos \theta)$$

$$\partial_r V_{(i)} = \sum_l A_l l r^{l-1} P_l(\cos \theta)$$

Thus Eq. (3) gives (as $\sigma_f = 0$) $\epsilon \partial_r V_{(o)}|_{r=R} = \epsilon_0 \partial_r V_{(i)}|_{r=R}$

$$\Rightarrow \epsilon \left[-E_0 P_1 - \sum_l \frac{B_l(l+1)}{R^{l+2}} P_l \right] = \epsilon_0 \sum_l A_l l R^{l-1} P_l$$

Equating coefficients of P_l for each l gives

$$l=1: \quad \epsilon \left(-E_0 - \frac{2B_1}{R^3} \right) = \epsilon_0 A_1$$

$$l \neq 1: \quad -\epsilon \frac{B_l(l+1)}{R^{l+2}} = \epsilon_0 A_l l R^{l-1}$$

Using (x) the $l=1$ equation gives

$$-\epsilon \left[E_0 + \frac{2}{R^3} R^3 (A_1 + E_0) \right] = \epsilon_0 A_1$$

$$\Rightarrow \underline{A_1 = -\frac{3\epsilon}{\epsilon_0 + 2\epsilon} E_0} \Rightarrow \underline{B_1 = \frac{\epsilon_0 - \epsilon}{\epsilon_0 + 2\epsilon} E_0 R^3}$$

Using (xx) the $l \neq 1$ equation gives

$$-\epsilon \frac{R^{2l+1} A_l(l+1)}{R^{l+2}} = \epsilon_0 A_l l R^{l-1} \Rightarrow \underline{A_l = 0} \Rightarrow \underline{B_l = 0}$$

$$\Rightarrow \underline{V_{\text{outside}}(\vec{r}) = -E_0 r \cos \theta - \frac{\epsilon - \epsilon_0}{2\epsilon + \epsilon_0} \frac{E_0 R^3}{r^2} \cos \theta}$$

(The first term is due to the external field, the second term is a dipole potential)

$$\underline{\underline{V_{\text{inside}}(\vec{r}) = -\frac{3\epsilon}{\epsilon_0 + 2\epsilon} E_0 r \cos \theta}}$$

As $r \cos \theta = z$, we evaluate $\vec{E}_{\text{inside}} = -\nabla V_{\text{inside}}$ using cartesian coordinates

$$\Rightarrow \underline{\underline{\vec{E}_{\text{inside}}(\vec{r}) = \frac{3\epsilon}{\epsilon_0 + 2\epsilon} E_0 \hat{z} \quad (\text{constant})}}$$

(Introducing the dielectric constant $\kappa \equiv \epsilon/\epsilon_0 (> 1)$, we see that

$$E_{\text{inside}} = \frac{3\kappa}{1 + 2\kappa} E_0 = \frac{3}{1/\kappa + 2} E_0 > E_0)$$

Problem 2

(a) Eq. (5) is a statement of conservation of energy for a volume Ω bounded by a surface a , inside of which there can be electric charges (also known as Poynting's theorem).

U_{em} is the energy stored in the electromagnetic fields inside Ω . Thus dU_{em}/dt is the rate of change of this energy.

$\oint \vec{S} \cdot d\vec{a}$ is the rate at which energy is carried out of Ω (by passing out through the surface a) by the electromagnetic fields (here $\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})$ is the Poynting vector)

dW/dt is the rate at which work is done on the charges inside Ω by the fields (really just by $\vec{E}$, since $\vec{B}$ does no work)

In words, the electromagnetic energy inside Ω can decrease due to energy flowing out and due to work being done on the charges (which increases their mechanical (kinetic) energy).

$$(b) \quad \vec{j} = \sigma \vec{E} \Rightarrow \vec{E} = \frac{\vec{j}}{\sigma} = \frac{I}{\sigma \pi b^2} \hat{z}$$

To find $\vec{B}$ we use Ampere's law on integral form on a circular loop $s = \text{constant}$. $\vec{B}$ will be tangential to this loop, with a constant magnitude B and a direction given by a right-hand rule:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{inside}} \Rightarrow B \cdot 2\pi s = \mu_0 j \pi s^2$$

$$\Rightarrow B = \frac{\mu_0 j s}{2} = \frac{\mu_0 I}{2\pi b} \frac{s}{b} \Rightarrow \underline{\underline{\vec{B} = \frac{\mu_0 I}{2\pi b} \left(\frac{s}{b}\right) \hat{\phi}}}$$

$$(c) \quad \vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{1}{\cancel{\mu_0}} \frac{I}{\sigma \pi b^2} \frac{\cancel{\mu_0} I s}{2\pi b^2} \underbrace{(\hat{z} \times \hat{\phi})}_{-\hat{s}} = -\frac{I^2 s}{2\pi^2 \sigma b^4} \hat{s}$$

$\Rightarrow \vec{S}$ points radially, towards the wire axis
 Thus $\oint_a \vec{S} \cdot d\vec{a}$ gets no contribution from the flat (top & bottom) surfaces of a , as $\vec{S} \cdot \hat{n} = 0$ there
 Thus the only contribution is from the curved surface where $s = b$ and $\hat{n} = \hat{s}$

$$\Rightarrow \oint_a \vec{S} \cdot d\vec{a} = -L \cdot \cancel{2\pi b} \cdot \frac{I^2}{\cancel{2\pi} \sigma b^{\cancel{2}2}} = -\frac{L}{\sigma \pi b^2} \cdot I^2$$

As the fields are static, $dU_{\text{em}}/dt = 0 \Rightarrow \underline{\underline{\frac{dW}{dt} = \frac{L}{\sigma \pi b^2} I^2}}}$

This equals RI^2 (power loss due to ohmic heating),
 as $V = RI = EL \Rightarrow R = EL/I = L/(\sigma \pi b^2)$

Problem 3.

(a) Due to the spherical symmetry of the charge distribution, it is natural to introduce spherical coordinates (r, θ, ϕ) , with ρ independent of θ and ϕ . Also, ρ is clearly only nonzero on the shell, i.e. for $r = R(t)$. This suggests that ρ can be written as $\rho = f(r, t) \delta(r - R(t))$ where the function $f(r, t)$ is to be determined. Actually, due to the delta function which requires $r = R(t)$ to be nonzero, we can write $f(r, t) \delta(r - R(t)) = f(R(t), t) \delta(r - R(t)) \equiv g(t) \delta(r - R(t))$ where $g(t) = f(R(t), t)$. To determine $g(t)$ we use that the total charge Q is conserved:

$$Q = \int d^3r \rho(\vec{r}, t) = \int_0^{2\pi} d\phi \int_{-1}^1 d(\cos\theta) \int_0^\infty dr r^2 g(t) \delta(r - R(t))$$

$$= 2\pi \cdot 2 \cdot g(t) \cdot R^2(t) \Rightarrow g(t) = \frac{Q}{4\pi R^2(t)}$$

$$\Rightarrow \rho(\vec{r}, t) = \frac{Q}{4\pi R^2(t)} \delta(r - R(t))$$

$$\Rightarrow \underline{\underline{\rho(\vec{r}, t) = \frac{Q}{4\pi r^2} \delta(r - R(t))}} \quad (\text{as } \delta(\cdot) = 0 \text{ for } r \neq R(t))$$

(b) We can write $\vec{E} = E_r \hat{r} + E_\theta \hat{\theta} + E_\phi \hat{\phi}$. Due to the spherical symmetry of the problem, E_θ and E_ϕ must be 0, and the only remaining nonzero component E_r must be independent of θ and ϕ . Therefore $\vec{E}(\vec{r}, t) = E_r(r, t) \hat{r}$. It then follows from the formula for the curl in spherical coordinates that $\underline{\underline{\nabla \times \vec{E} = 0}}$

Thus we have $\nabla \cdot \vec{E} = \rho/\epsilon_0$ (Gauss's law)
 $\nabla \times \vec{E} = 0$

The 2nd equation implies that this set of equations for the spatial structure of $\vec{E}$ takes the same form as for an electrostatics problem (the time t is mathematically just an "innocuous" parameter in ρ that will make $\vec{E}$ time-dependent). Given the spherical symmetry it is then natural to solve the problem using Gauss's law in integral form obtained from integrating $\nabla \cdot \vec{E} = \rho/\epsilon_0$ over a volume Ω enclosed by a "Gaussian" surface a and then using the divergence theorem (D.T.):

$$\int_{\Omega} d^3r \nabla \cdot \vec{E} \stackrel{\text{D.T.}}{=} \oint_a d\vec{a} \cdot \vec{E} = \frac{1}{\epsilon_0} \int_{\Omega} d^3r \rho = \frac{Q_{\text{inside}}}{\epsilon_0}$$

At this stage we exploit the symmetry by taking the Gaussian surface to be spherical (with radius r) and centered at the origin, as E_r is constant on this surface. Thus

$$\epsilon_0 \oint_a \vec{E} \cdot d\vec{a} = \epsilon_0 E_r 4\pi r^2 = Q_{\text{inside}} = \begin{cases} Q & \text{if } r > R(t) \\ 0 & \text{if } r < R(t) \end{cases}$$
$$= Q \Theta(r - R(t)) \Rightarrow \underline{\underline{\vec{E}(\vec{r}, t) = \frac{Q}{4\pi\epsilon_0 r^2} \Theta(r - R(t)) \hat{r}}}$$

Here $\Theta(u)$ is the (Heaviside) step function

$$\Theta(u) \equiv \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u < 0 \end{cases}$$

(not to be confused with the angle θ !)

$$(c) \quad \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t} \frac{Q}{4\pi r^2} \delta(r - R(t)) = \frac{Q}{4\pi r^2} \frac{\partial}{\partial t} \delta(r - R(t))$$

Define $u = r - R(t)$ and use the chain rule:

$$\frac{\partial}{\partial t} \delta(u) = \frac{\partial u}{\partial t} \frac{d}{du} \delta(u) = -\dot{R}(t) \frac{d}{du} \delta(u)$$

$$\Rightarrow \underline{\frac{\partial \rho}{\partial t} = -\frac{Q \dot{R}(t)}{4\pi r^2} \frac{d}{du} \delta(u)}$$

As the spatial dependence of $\vec{j}(\vec{r}, t)$ takes the form $j_r \hat{r}$, we see from the formula set that

$$\begin{aligned} \nabla \cdot \vec{j} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 j_r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{Q \dot{R}(t)}{4\pi} \delta(r - R(t)) \right) \\ &= \frac{Q \dot{R}(t)}{4\pi r^2} \frac{\partial}{\partial r} \delta(r - R(t)) \end{aligned}$$

$$\frac{\partial}{\partial r} \delta(u) = \frac{\partial u}{\partial r} \frac{d}{du} \delta(u) = 1 \cdot \frac{d}{du} \delta(u) = \frac{d}{du} \delta(u)$$

$$\Rightarrow \underline{\nabla \cdot \vec{j} = \frac{Q \dot{R}(t)}{4\pi r^2} \frac{d}{du} \delta(u)}$$

Comparing the underlined expressions for $\frac{\partial \rho}{\partial t}$ and $\nabla \cdot \vec{j}$ shows that the continuity equation is satisfied

(d) Because of the spherical symmetry (the current density $\vec{j}(\vec{r}, t) = j_r(r, t) \hat{r}$) it follows that $\vec{B}$ must take the form $\vec{B}(\vec{r}, t) = B_r(r, t) \hat{r}$. Thus the formula for the curl in spherical coordinates gives $\underline{\nabla \times \vec{B} = 0}$. Alternatively, this result can also be shown from the Ampere-Maxwell law

$$\nabla \times \vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

We know $\vec{j}$, so it remains to find $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$:

$$\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \epsilon_0 \frac{\partial}{\partial t} \frac{Q}{4\pi\epsilon_0 r^2} \theta(r - R(t)) \hat{r}$$

$$= \frac{Q \hat{r}}{4\pi r^2} \frac{\partial}{\partial t} \theta(r - R(t)), \quad \text{where}$$

$$\frac{\partial}{\partial t} \theta(r - R(t)) = \frac{\partial u}{\partial t} \frac{d}{du} \theta(u) = -\dot{R}(t) \delta(u)$$

$$\Rightarrow \epsilon_0 \frac{\partial \vec{E}}{\partial t} = -\frac{Q \dot{R}(t)}{4\pi r^2} \delta(r - R(t)) \hat{r} = -\vec{j} \Rightarrow \nabla \times \vec{B} = 0$$

Thus $\vec{B}$ obeys $\nabla \cdot \vec{B} = 0$ (Gauss's law for $\vec{B}$)
 $\nabla \times \vec{B} = 0$

These equations take the same form as those for $\vec{E}$, with ρ replaced by 0 ($= \rho_{\text{magnetic}}$). Using the same method to find $\vec{B}$ thus gives

$$\underline{\underline{\vec{B}(\vec{r}, t) = 0}}$$

(e) To consider radiation we consider a spherical surface a of radius r centered at the origin. The time-average of the energy passing through the surface per unit time is $P(r) = \oint_a d\vec{a} \cdot \langle \vec{S} \rangle$

where $\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})$ and $\langle \vec{S} \rangle$ its time average. The shell radiates if $\lim_{r \rightarrow \infty} P(r) \neq 0$.

Since $\vec{B} = 0$, $\vec{S} = \vec{0}$, so there is no radiation. (Alternatively, the conclusion of no radiation can also be argued from $\vec{E} \parallel \hat{r} \Rightarrow \vec{S} \perp \hat{r} \Rightarrow \vec{S} \cdot d\vec{a} = 0$. The fact that $|\vec{E}| \propto 1/r^2$ is also inconsistent with radiation.