TFY4240 Electromagnetic theory: Solution to exam, May 2018

Problem 1

(a) The problem is to find the potential V in some region Ω of a physical system, given (i) the charge density in Ω and (ii) boundary conditions for V on the boundary of Ω. The method consists of constructing an alternative system where (i) and (ii) are unchanged, and solving the problem for the alternative system instead (because a uniqueness theorem guarantees that the solution for V in Ω will be the same in both systems). The alternative system contains "image" charges in the region outside of Ω, hence the name of the method.

(b) Using $\nabla \cdot \boldsymbol{D} = \rho_f$, $\boldsymbol{D} = \epsilon \boldsymbol{E}$, and $\boldsymbol{E} = -\nabla V$ leads to the Poisson equation

$$\nabla^2 V = -\frac{\rho_f}{\epsilon} \tag{1}$$

We will solve this in each of the two regions and then match the solutions at the boundary using the boundary conditions. To find the potential in any one of the two regions, we may modify the charge distribution in the other region. To find the potential in region 1, we remove medium 2, letting all space be filled by medium 1, and put an image charge q_1 at $z = d$ (and, by symmetry, we take the x and y positions of the image charge to the $x = 0$, $y = 0$). Using also that $\rho_f = 0$ in region 1, the potential in region 1 is

$$V_1(x, y, z) = \frac{1}{4\pi\epsilon_1} \frac{q_1}{\sqrt{x^2 + y^2 + (z-d)^2}}. \tag{2}$$

To find the potential in region 2, we remove medium 1, let all space be filled by medium 2, and put an image charge q_2 at $z = -d$. Using also that $\rho_f = q\delta(\boldsymbol{r} - d\hat{\boldsymbol{z}})$ in region 2 then gives

$$V_2(x, y, z) = \frac{1}{4\pi\epsilon_2} \left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{q_2}{\sqrt{x^2 + y^2 + (z+d)^2}} \right]. \tag{3}$$

Now we invoke the boundary conditions (BC), which are (cf. Eq. (1) in the list of formulas)

$$V_2 - V_1 = 0, \tag{4}$$

$$\epsilon_2 \partial_z V_2 - \epsilon_1 \partial_z V_1 = -\sigma_f, \tag{5}$$

where all quantities are evaluated at $z = 0$. The first BC gives, after cancelling common factors,

$$\frac{q_1}{\epsilon_1} = \frac{q + q_2}{\epsilon_2}. \tag{6}$$

For the second BC we need to evaluate

$$\frac{\partial}{\partial z} [x^2 + y^2 + (z-a)^2]^{-1/2} \Big|_{z=0} = -\frac{1}{2} [x^2 + y^2 + (z-a)^2]^{-3/2} \cdot 2(z-a) \cdot 1 \Big|_{z=0} = \frac{a}{[x^2 + y^2 + a^2]^{3/2}}. \tag{7}$$

The second BC then gives, upon cancelling some common factors and using that $\sigma_f = 0$ (because there is no free surface charge in the system),

$$qd + q_2(-d) - q_1 d = 0 \quad \Rightarrow \quad q - q_2 = q_1. \tag{8}$$

Inserting this into the first BC gives

$$\frac{q - q_2}{\epsilon_1} = \frac{q + q_2}{\epsilon_2} \quad \Rightarrow \quad \epsilon_2(q - q_2) = \epsilon_1(q + q_2) \quad \Rightarrow \quad (\epsilon_1 + \epsilon_2) q_2 = (\epsilon_2 - \epsilon_1) q \quad \Rightarrow \quad q_2 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} q \tag{9}$$

and

$$q_1 = q - q_2 = \frac{q}{\epsilon_2 + \epsilon_1} [\epsilon_2 + \epsilon_1 - (\epsilon_2 - \epsilon_1)] = \frac{2\epsilon_1}{\epsilon_2 + \epsilon_1} q. \tag{10}$$

Inserting these expressions for q_1 and q_2 back into V_1 and V_2 concludes the solution of the problem.

(c) As the charge q is in region 2, we must use V_2 to find the force. The first term in V_2 (proportional to q) is due to q itself, and since a particle doesn't act with a force on itself, the force on q is given solely from the electric field due to the second term $\propto$ the image charge q_2. As the potential is of Coulomb type, so is the field and the force, so it may be calculated simply as

$$\boldsymbol{F} = \frac{q q_2}{4\pi\epsilon_2 R_{q_2}^2} \hat{\boldsymbol{R}}_{q_2}, \tag{11}$$

where R_{q_2} is the distance between q and q_2, which is $2d$, and $\hat{\boldsymbol{R}}_{q_2}$ is the unit vector pointing from q_2 to q, which is $\hat{\boldsymbol{z}}$. Inserting also for q_2 gives

$$\boldsymbol{F} = \frac{q^2}{4\pi\epsilon_2} \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} \frac{1}{4d^2} \hat{\boldsymbol{z}}. \tag{12}$$

Of course this result could also have been found by a more explicit calculation.

For the special case $\epsilon_1 = \epsilon_2$, the force $\boldsymbol{F}$ is seen to vanish. This is reasonable, as in this case there is only a single medium filling all space, so there is no interface and thus no surface bound charge that can give a force on q. (Remark: Arguing that $\boldsymbol{F} = 0$ is reasonable because $q_2 = 0$ is not a good answer, because it refers to the image charge q_2 which is fictitious, not a part of the actual physical system.)

(d) The volume bound charge density in a simple dielectric medium is

$$\rho_b = -\nabla \cdot \boldsymbol{P} = -\epsilon_0 \chi \nabla \cdot \boldsymbol{E} = -\epsilon_0 \chi \frac{1}{\epsilon} \nabla \cdot \boldsymbol{D} = -\epsilon_0 \chi \frac{1}{\epsilon} \rho_f = -\frac{1}{\kappa}(\kappa - 1)\rho_f = (1/\kappa - 1)\rho_f, \tag{13}$$

where I introduced the dielectric constant $\kappa = \epsilon/\epsilon_0$ and used $\chi = \kappa - 1$. This shows that there is volume bound charge only where there is volume free charge. Applying this result to each of the two regions thus gives

$$\rho_{b,1} = (1/\kappa_1 - 1)\rho_{f,1} = 0, \tag{14}$$

$$\rho_{b,2} = (1/\kappa_2 - 1)\rho_{f,2} = (1/\kappa_2 - 1) q \delta(\boldsymbol{r} - d\hat{\boldsymbol{z}}). \tag{15}$$

This result can be summarized as

$$\rho_b = (1/\kappa_2 - 1) q \delta(\boldsymbol{r} - d\hat{\boldsymbol{z}}). \tag{16}$$

Alternatively, one may find ρ_b using a more explicit calculation (but this is more technical so I don't really recommend it): Consider a contribution $V_Q = \frac{Q}{4\pi\epsilon R_Q}$ to the potential V in a given region, where $\boldsymbol{R}_Q = \boldsymbol{r} - \boldsymbol{r}_Q$. The corresponding electric field and polarization is

$$\boldsymbol{E}_Q = -\nabla V_Q = -\frac{Q}{4\pi\epsilon} \underbrace{\nabla(1/R_Q)}_{-\hat{R}_Q/R_Q^2} = \frac{Q}{4\pi\epsilon} \frac{\hat{R}_Q}{R_Q^2} \quad \Rightarrow \quad \boldsymbol{P}_Q = \epsilon_0 \chi \boldsymbol{E}_Q = \frac{Q}{4\pi} \frac{\kappa - 1}{\kappa} \frac{\hat{R}_Q}{R_Q^2} \tag{17}$$

and the corresponding contribution to ρ_b is

$$\rho_{b,Q} = -\nabla \cdot \boldsymbol{P}_Q = -\frac{Q}{4\pi} \frac{\kappa - 1}{\kappa} \underbrace{\nabla \cdot \frac{\hat{R}_Q}{R_Q^2}}_{4\pi\delta(\boldsymbol{R}_Q)} = (1/\kappa - 1) Q \delta(\boldsymbol{r} - \boldsymbol{r}_Q). \tag{18}$$

If $\boldsymbol{r}_Q$ is outside the region in which V_Q is a valid contribution to V, $\boldsymbol{r} \neq \boldsymbol{r}_Q$ for any $\boldsymbol{r}$ in the region, so the delta function will always be zero. Thus image charges do not contribute to ρ_b. The only contribution is therefore from the point charge q, giving again Eq. (16).

(e) A dielectric body/region contributes a surface bound charge density $\boldsymbol{P} \cdot \boldsymbol{n}$ where $\boldsymbol{P}$ is the polarization and $\boldsymbol{n}$ is a unit vector perpendicular to the surface, pointing out of the body/region. Since in our case we have an interface between two regions of different dielectric media, each region contributes a surface bound charge density, giving the total surface bound charge density

$$\begin{aligned} \sigma_b &= \sigma_{b,1} + \sigma_{b,2} = \boldsymbol{P}_1 \cdot (+\hat{\boldsymbol{z}}) + \boldsymbol{P}_2 \cdot (-\hat{\boldsymbol{z}}) = (\boldsymbol{P}_1 - \boldsymbol{P}_2) \cdot \hat{\boldsymbol{z}} = P_{1z} - P_{2z} \\ &= \epsilon_0(\chi_1 E_{1z} - \chi_2 E_{2z}) = \epsilon_0(\chi_2 \partial_z V_2 - \chi_1 \partial_z V_1). \end{aligned} \tag{19}$$

Using (7) gives

$$\sigma_b(x,y) = \epsilon_0 \left[\frac{\chi_2}{\epsilon_2}(q - q_2) - \frac{\chi_1}{\epsilon_1} q_1 \right] \frac{d}{4\pi[x^2 + y^2 + d^2]^{3/2}}. \tag{20}$$

Inserting for q_2 and q_1, the expression in the square brackets is

$$\begin{aligned} q_1 \left(\frac{\chi_2}{\epsilon_2} - \frac{\chi_1}{\epsilon_1} \right) &= \frac{q}{\epsilon_0} \frac{2\epsilon_1}{\epsilon_1 + \epsilon_2} \left(\frac{\kappa_2 - 1}{\kappa_2} - \frac{\kappa_1 - 1}{\kappa_1} \right) = \frac{q}{\epsilon_0} \frac{2\kappa_1}{\kappa_1 + \kappa_2} \frac{\kappa_1(\kappa_2 - 1) - \kappa_2(\kappa_1 - 1)}{\kappa_1 \kappa_2} \\ &= \frac{2q}{\epsilon_2} \frac{\kappa_2 - \kappa_1}{\kappa_1 + \kappa_2}, \end{aligned} \tag{21}$$

giving

$$\sigma_b(x,y) = \frac{1}{2\pi\kappa_2} \frac{\kappa_2 - \kappa_1}{\kappa_1 + \kappa_2} \frac{qd}{[x^2 + y^2 + d^2]^{3/2}}. \tag{22}$$

Problem 2

(a) The motion of the bar implies that the horizontal position x of the bar changes, so the flux Φ enclosed by the circuit changes. By the flux rule, an emf $\varepsilon = -d\Phi/dt$ is generated in the circuit. The current is given by Ohm's law, giving $I = |\varepsilon|/R$ (only the bar contributes to the resistance, as the rails are perfect conductors), i.e.

$$I = \frac{1}{R} \left| \frac{d}{dt}(B\ell x) \right| = \frac{B\ell v}{R}. \tag{23}$$

The direction of the current can be found in various ways:

- From the Lorentz force $\boldsymbol{F}_m = q\boldsymbol{v} \times \boldsymbol{B}$ on a positive charge q in the bar: $\boldsymbol{F}_m$ is directed upwards, corresponding to a counterclockwise current.
- From Lenz's law: As the area of the circuit increases, the induced current will create a magnetic field whose direction is opposite to the external field $\boldsymbol{B}$, i.e. out of the page. By a right-hand rule, the current is thus counterclockwise.
- From an alternative calculation of I that also includes the sign: The flux through the circuit is $\Phi = \int_a \boldsymbol{B} \cdot d\boldsymbol{a}$ where a is a surface having the circuit as its boundary. Obviously the simplest choice for a is the flat rectangle bounded by the circuit. Picking the positive circulation direction to be clockwise, it follows by a right-hand rule that the unit vector $\boldsymbol{n}$ in $d\boldsymbol{a} = \boldsymbol{n} da$ is into the page, so that $\boldsymbol{B} \cdot d\boldsymbol{a}$ is positive, giving $\Phi = +BLx$. Thus $I = \varepsilon/R = -\frac{1}{R}\frac{d}{dt}(BLx) = -\frac{BLv}{R}$. Since the sign of I is negative, the current is against the positive circulation direction, i.e. the current is counterclockwise.

(b) Due to the current I in the bar, a force $\boldsymbol{F} = I\boldsymbol{L} \times \boldsymbol{B}$ acts on it. Here $\boldsymbol{L}$ is a vector of length L in the direction of the current in the bar, i.e. upwards. It follows that $\boldsymbol{F}$ is to the left, so it opposes the bar's motion to the right. Newton's 2nd law (N2) for the bar in the horizontal direction is thus

$$m dv/dt = -ILB. \tag{24}$$

Using $v = RI/(BL)$ and inserting this into our N2 to eliminate v gives

$$\frac{mR}{BL} dI/dt = -BLI \quad \Rightarrow \quad \frac{dI}{dt} = -\alpha I \quad \text{where } \alpha = \frac{B^2 L^2}{mR}. \tag{25}$$

The solution of this differential equation is $I = I_0 \exp(-\alpha t)$ where I_0 is given by inserting the initial velocity v_0 into the relationship between I and v, giving $I_0 = \frac{BLv_0}{R}$.

(d) The total energy dissipated as Joule heating between $t = 0$ and $t = \infty$ is

$$\begin{aligned} \int_0^\infty P(t)dt &= \int_0^\infty dt\, RI^2(t) = RI_0^2 \int_0^\infty dt\, \exp(-2\alpha t) = RI_0^2 \frac{1}{-2\alpha} \exp(-2\alpha t) \Big|_0^\infty = \frac{RI_0^2}{2\alpha} \\ &= \frac{R}{2} \frac{mR}{B^2 L^2} \frac{B^2 L^2 v_0^2}{R^2} = \frac{1}{2} m v_0^2. \end{aligned} \tag{26}$$

This shows that the entire initial kinetic energy of the bar is transformed into heat. Also note that the external field B is constant throughout, so the energy stored in it does not change with time. Thus the result (26) is a statement of energy conservation.

Problem 3

(a) The Maxwell equations for magnetostatics are

$$\nabla \cdot \boldsymbol{B} = 0, \tag{27}$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}. \tag{28}$$

The introduction of the vector potential $\boldsymbol{A}$ via $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is directly motivated by Eq. (27), because it then becomes $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$, which is the identity (9) in the formula set, i.e. it is automatically satisfied for *any* $\boldsymbol{A}$, and thus it does not put any constraint on $\boldsymbol{A}$. Therefore we are left with Eq. (28), which upon using identity (11) becomes the differential equation

$$\nabla^2 \boldsymbol{A} - \nabla(\nabla \cdot \boldsymbol{A}) = -\mu_0 \boldsymbol{j}. \tag{29}$$

(b) (i) Under the gauge transformation, $\boldsymbol{B}$ changes to

$$\boldsymbol{B}' = \nabla \times \boldsymbol{A}' = \nabla \times \boldsymbol{A} + \nabla \times \nabla\lambda = \boldsymbol{B} + 0 = \boldsymbol{B}, \tag{30}$$

where we used identity (10) in the formula set. (ii) The fact that $\boldsymbol{B}$ is invariant under a gauge transformation implies that we have some freedom in choosing $\boldsymbol{A}$, and this freedom can be implemented as the Coulomb gauge condition $\nabla \cdot \boldsymbol{A} = 0$. (To see this, suppose $\boldsymbol{B} = \nabla \times \boldsymbol{A}_0$ where $\boldsymbol{A}_0$ does not satisfy the Coulomb gauge condition. We can then do a gauge transformation $\boldsymbol{A} = \boldsymbol{A}_0 + \nabla\lambda$. Imposing the Coulomb gauge condition on $\boldsymbol{A}$ thus gives $\nabla^2\lambda = -\nabla \cdot \boldsymbol{A}_0$, which is a Poisson equation for the unknown function λ.)

In the Coulomb gauge, the differential equation (29) reduces to $\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{j}$, which is a Poisson equation for each cartesian component. Thus by analogy with the Poisson equation for V we have the correspondence $V \to A_i$, $\rho/\epsilon_0 \to \mu_0 j_i$. The given solution for V then shows that the solution for $\boldsymbol{A}$ is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{31}$$

(c) From the general formula set, the electromagnetic field momentum is given by

$$\boldsymbol{P}_{\rm EM} = \epsilon_0 \int d^3r \, (\boldsymbol{E} \times \boldsymbol{B}) = \epsilon_0 \int d^3r \, (\boldsymbol{E} \times (\nabla \times \boldsymbol{A})), \tag{32}$$

where the integral is over all space. To show that $\boldsymbol{P}_{\rm EM} = \int d^3r \, \rho(\boldsymbol{r})\boldsymbol{A}(\boldsymbol{r})$, we start by looking at the cartesian components of $\boldsymbol{P}_{\rm EM}$. Thus consider

$$\begin{aligned}(\boldsymbol{E} \times (\nabla \times \boldsymbol{A}))_i &= \epsilon_{ijk} E_j (\nabla \times \boldsymbol{A})_k = \epsilon_{ijk} E_j \epsilon_{klm} \partial_l A_m = \epsilon_{kij} \epsilon_{klm} E_j \partial_l A_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) E_j \partial_l A_m = E_j \partial_i A_j - E_j \partial_j A_i.\end{aligned} \tag{33}$$

Putting this back in the integral, using integration by parts on each term, and throwing away the boundary terms at infinity (as stated in the hint), gives

$$P_{\rm EM,i} = \epsilon_0 \int d^3r \, (-A_j \partial_i E_j + A_i \partial_j E_j). \tag{34}$$

In the first term we use $E_j = -\partial_j V$ and the fact that partial derivatives commute to write $A_j \partial_i \partial_j V = A_j \partial_j \partial_i V$. In the second term we note that $\partial_j E_j = \nabla \cdot \boldsymbol{E}$. Thus

$$P_{\rm EM,i} = \epsilon_0 \int d^3r \, \left[A_j \partial_j(\partial_i V) + A_i \nabla \cdot \boldsymbol{E}\right]. \tag{35}$$

In the first term we do another integration by parts, which effectively turns that term in the integrand into $-(\partial_i V)\partial_j A_j = E_i \nabla \cdot \boldsymbol{A} = 0$, where we used the Coulomb gauge condition $\nabla \cdot \boldsymbol{A} = 0$. In the second term we use Gauss' law $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$. This gives $P_{\mathrm{EM,i}} = \int d^3r\, \rho(\boldsymbol{r}) A_i(\boldsymbol{r})$, which is the ith component of the desired result.

Remark: This result does NOT imply that the integrands in the two expressions for $\boldsymbol{P}_{\mathrm{EM}}$ are equal. In particular, $\rho \boldsymbol{A}$ is nonzero only at points where $\rho \neq 0$, which may be a very limited set of points, as illustrated in the example in (d).

(d) The two point charges don't contribute to $\boldsymbol{A}$, since the only source of $\boldsymbol{A}$ is current density $\boldsymbol{j}$, cf. Eq. (31). Also, the magnetic dipole contributes nothing to ρ. Thus we have for this system

$$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r} - d\hat{\boldsymbol{z}}) + (-q)\delta(\boldsymbol{r} - (-d\hat{\boldsymbol{z}})), \tag{36}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}, \tag{37}$$

where we used Eq. (7) in the formula set and the fact that $\boldsymbol{r}_0$ is the origin. Inserting these expressions into the integral and evaluating it gives

$$\boldsymbol{P}_{\mathrm{EM}} = q\left[\boldsymbol{A}(d\hat{\boldsymbol{z}}) - \boldsymbol{A}(-d\hat{\boldsymbol{z}})\right] = \frac{q\mu_0}{2\pi} \frac{\boldsymbol{m} \times \hat{\boldsymbol{z}}}{d^2}. \tag{38}$$

(e) (i) From the formula set, one can write $\boldsymbol{P}_{\mathrm{EM}} = \int d^3r\, \boldsymbol{g}_{\mathrm{EM}}$ where

$$\boldsymbol{g}_{\mathrm{EM}} = \frac{\boldsymbol{S}}{c^2} \tag{39}$$

is the momentum density of the electromagnetic fields and $\boldsymbol{S}$ is the Poynting vector. (ii) The Poynting vector $\boldsymbol{S}$ can be interpreted as the energy per unit time per unit area transported by the electromagnetic fields. This means that $\boldsymbol{S}$ is an energy current density (energy flux density), similarly to how $\boldsymbol{j}$ is the (charge) current density. Thus $\oint_a \boldsymbol{S} \cdot \boldsymbol{da}$ is the amount of energy per unit time leaving a volume Ω through its surface a.