# TFY4240 Electromagnetic theory: Solution to exam, spring 2019

## Problem 1

In this problem we will encounter equations on the form

$$\sum_{\ell=0}^{\infty} f_\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} g_\ell P_\ell(\cos\theta), \tag{1}$$

where $f_\ell$ and $g_\ell$ are coefficients. Eq. (1) implies that[1]

$$f_\ell = g_\ell. \tag{2}$$

**(a)** Let us first consider the potential outside the shell ($r > R$), which we call $V^{\rm out}$. As there is no charge there, $V^{\rm out}$ obeys the Laplace equation. Since furthermore the potential **on** the shell is independent of the azimuthal angle $\phi$, the same will be true for $V^{\rm out}$, which therefore may be expanded as (cf. Eq. (1) in the set of specific formulas)

$$V^{\rm out}(r,\theta) = \sum_{\ell=0}^{\infty} \left( A_\ell^{\rm out} r^\ell + \frac{B_\ell^{\rm out}}{r^{\ell+1}} \right) P_\ell(\cos\theta). \tag{3}$$

Since there is no charge at infinity (i.e. for $r \to \infty$), we may set $V^{\rm out} = 0$ there. This gives $A_\ell^{\rm out} = 0$ for all $\ell$. Furthermore, the condition that $V$ is a continuous function gives for $r = R$ that $V^{\rm out}(R,\theta) = V(R,\theta)$, i.e.

$$\sum_{\ell=0}^{\infty} \frac{B_\ell^{\rm out}}{R^{\ell+1}} P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} V_\ell P_\ell(\cos\theta) \tag{4}$$

which by (1)-(2) implies $B_\ell^{\rm out} = V_\ell R^{\ell+1}$. Thus

$$V^{\rm out}(r,\theta) = \sum_{\ell=0}^{\infty} V_\ell \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta). \tag{5}$$

Let us next consider the potential inside the shell ($r < R$), which we call $V^{\rm in}$. By the same arguments as for $V^{\rm out}$, we may expand $V^{\rm in}$ as

$$V^{\rm in}(r,\theta) = \sum_{\ell=0}^{\infty} \left( A_\ell^{\rm in} r^\ell + \frac{B_\ell^{\rm in}}{r^{\ell+1}} \right) P_\ell(\cos\theta). \tag{6}$$

As there is no charge at the origin ($r = 0$), $V^{\rm in}$ cannot diverge as $r \to 0$. This implies $B_\ell^{\rm in} = 0$ for all $\ell$. Furthermore, the condition that $V$ is continuous gives for $r = R$ that $V^{\rm in}(R,\theta) = V(R,\theta)$, i.e.

$$\sum_{\ell=0}^{\infty} A_\ell^{\rm in} R^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} V_\ell P_\ell(\cos\theta), \tag{7}$$

which implies $A_\ell^{\rm in} = V_\ell / R^\ell$. Thus

$$V^{\rm in}(r,\theta) = \sum_{\ell=0}^{\infty} V_\ell \left(\frac{r}{R}\right)^{\ell} P_\ell(\cos\theta). \tag{8}$$

**(b)** Using Eq. (4) in the set of specific formulas (which may be derived from Gauss's law), and that the

[1] You could assume (2) without proof. To prove it, multiply (1) by $P_{\ell'}(\cos\theta)\sin\theta$ and integrate $\theta$ from 0 to $\pi$, change variables to $x = \cos\theta$, and use Eq. (3) in the specific formula set.

spherical shape of the shell implies $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$ , i.e. $\partial/\partial n = \partial/\partial r$, the surface charge density is

$$\begin{aligned}
\sigma &= -\epsilon_0 \left[\frac{\partial V^{\text{out}}}{\partial r} - \frac{\partial V^{\text{in}}}{\partial r}\right]\Bigg|_{r=R} && (9)\\
&= -\epsilon_0 \sum_{\ell=0}^{\infty} V_\ell P_\ell(\cos\theta)\left[R^{\ell+1}\frac{\partial}{\partial r}r^{-(\ell+1)} - R^{-\ell}\frac{\partial}{\partial r}r^{\ell}\right]\Bigg|_{r=R} && (10)\\
&= -\epsilon_0 \sum_{\ell=0}^{\infty} V_\ell P_\ell(\cos\theta)\left[R^{\ell+1}(-1)(\ell+1)r^{-(\ell+1)-1} - R^{-\ell}\ell r^{\ell-1}\right]\Bigg|_{r=R} && (11)\\
&= -\epsilon_0 \sum_{\ell=0}^{\infty} V_\ell P_\ell(\cos\theta)\left[(-1)(\ell+1) - \ell\right]R^{-1} && (12)\\
&= \frac{\epsilon_0}{R}\sum_{\ell=0}^{\infty} V_\ell(2\ell+1)P_\ell(\cos\theta). && (13)
\end{aligned}$$

**(c) + (d)** The problem text asks for the electric potential produced by the polarization $\boldsymbol{P}$. I will call this potential $V_P$, to emphasize that it is not equal to the **total** electric potential (which would also include the potential of the external field that creates $\boldsymbol{P}$ in the first place). The associated electric field is $\boldsymbol{E}_P = -\nabla V_P$.

In general, $V_P(\boldsymbol{r})$ is the sum of two contributions: one from the volume bound charge density $\rho_b = -\nabla\cdot\boldsymbol{P}$, and one from the surface bound charge density $\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}}$. These two contributions may in principle be found from the same integration formulas that give the potential produced by any electrostatic volume or surface charge density, i.e.

$$\frac{1}{4\pi\epsilon_0}\int_\Omega \frac{\rho_b(\boldsymbol{r}')d^3r'}{|\boldsymbol{r}-\boldsymbol{r}'|} \quad \text{gives the contribution from the volume bound charge,} \tag{14}$$

$$\frac{1}{4\pi\epsilon_0}\int_a \frac{\sigma_b(\boldsymbol{r}')da'}{|\boldsymbol{r}-\boldsymbol{r}'|} \quad \text{gives the contribution from the surface bound charge.} \tag{15}$$

Here the integrals are over the volume $\Omega$ (more precisely, over the interior of the volume, excluding any surface contribution) and over the surface $a$, respectively, of the dielectric body. In our problem, since $\boldsymbol{P}$ is uniform, $\rho_b = -\nabla\cdot\boldsymbol{P} = 0$. So the only contribution to $V_P$ comes from the surface bound charge. Picking the $z$ axis in the direction of $\boldsymbol{P}$ and the origin at the center of the dielectric sphere, we have

$$\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} = P\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}} = P\cos\theta. \tag{16}$$

The fact that $\sigma_b$ is independent of $\phi$ means that we may make use of the mathematical similarities between this problem of finding $V_P$ and the problem of finding $V$ and $\sigma$ in (a) and (b) (this method is easier than trying to directly evaluate the integral in (15)). The result (13) may be written

$$\sigma = \sum_{\ell=0}^{\infty}\sigma_\ell P_\ell(\cos\theta), \tag{17}$$

where the coefficient $\sigma_\ell$ is

$$\sigma_\ell = \frac{\epsilon_0}{R}V_\ell(2\ell+1). \tag{18}$$

On the other hand, using that $P_1(\cos\theta) = \cos\theta$, Eq. (16) may be written

$$\sigma_b = PP_1(\cos\theta). \tag{19}$$

Equating (19) and (17) implies

$$\sigma_\ell = P\delta_{\ell,1}. \tag{20}$$

Next, we find $V_\ell$ from (18):

$$V_\ell = \frac{R}{\epsilon_0}\frac{\sigma_\ell}{2\ell+1} = \frac{R}{\epsilon_0}\frac{P}{2\ell+1}\delta_{\ell,1} = \frac{PR}{3\epsilon_0}\delta_{\ell,1}. \tag{21}$$

To find $V_P^{\rm out}$ we insert (21) in (5), which gives

$$V_P^{\rm out} = \sum_{\ell=0}^{\infty} \frac{PR}{3\epsilon_0}\delta_{\ell,1}\left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta) = \frac{PR^3\cos\theta}{3\epsilon_0 r^2}. \tag{22}$$

By introducing the electric dipole moment of the dielectric sphere, $\boldsymbol{p} = \int_\Omega \boldsymbol{P} = \frac{4\pi R^3}{3}\boldsymbol{P}$, we can rewrite $V_P^{\rm out}$ as

$$V_P^{\rm out} = \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{4\pi\epsilon_0 r^2}, \tag{23}$$

which we recognize as an electric dipole potential. Consequently, $\boldsymbol{E}_P^{\rm out}$ is an electric dipole field:

$$\boldsymbol{E}_P^{\rm out} = -\nabla V_P^{\rm out} = -\frac{\partial V_P^{\rm out}}{\partial r}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial V_P^{\rm out}}{\partial\theta}\hat{\boldsymbol{\theta}} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{24}$$

Note that for an electric dipole, the potential decays like $1/r^2$ and the field decays like $1/r^3$.

To find $V_P^{\rm in}$ we insert (21) in (8), which gives

$$V_P^{\rm in} = \sum_{\ell=0}^{\infty} \frac{PR}{3\epsilon_0}\delta_{\ell,1}\left(\frac{r}{R}\right)^{\ell} P_\ell(\cos\theta) = \frac{P}{3\epsilon_0} r\cos\theta = \frac{P}{3\epsilon_0} z. \tag{25}$$

Thus

$$\boldsymbol{E}_P^{\rm in} = -\nabla V_P^{\rm in} = -\frac{P}{3\epsilon_0}\hat{\boldsymbol{z}} = -\frac{\boldsymbol{P}}{3\epsilon_0}. \tag{26}$$

We see that $\boldsymbol{E}_P^{\rm in}$ is uniform and points in the direction opposite to $\boldsymbol{P}$.

Remarks about (c) and (d):

- Unfortunately some students assumed that the dielectric was simple (i.e. linear and isotropic). Note that the problem text did not say that the dielectric was simple, nor did it introduce any parameters characteristic of a simple dielectric, such as $\chi$, $\kappa$ $(= 1 + \chi)$, or $\epsilon$ $(= \epsilon_0\kappa)$. Furthermore, the equation $\boldsymbol{P} = \epsilon_0\chi\boldsymbol{E}$ valid for a simple dielectric was not in the set of specific formulas.
- Even if the dielectric had been simple, note that the equation $\boldsymbol{P} = \epsilon_0\chi\boldsymbol{E}$ relates the polarization $\boldsymbol{P}$ to the **total** electric field $\boldsymbol{E}$ inside a simple dielectric, not the electric field $\boldsymbol{E}_P^{\rm in}$ produced by the polarization. The two fields are related by $\boldsymbol{E} = \boldsymbol{E}_{\rm applied} + \boldsymbol{E}_P$, where $\boldsymbol{E}_{\rm applied}$ is the external field that the experimentalist applies. Note in particular that $\boldsymbol{E}$ and $\boldsymbol{E}_P^{\rm in}$ point in opposite directions.

## Problem 2

**(a)** 1. To find the potential in the region $z > 0$ we replace the conductor with an "image" point charge $q'$ at the point $(x, y, z) = (0, 0, -d)$. The potential is then

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z+d)^2}}\right]. \tag{27}$$

To find $q'$ we use the boundary condition on $V$, which is that $V = 0$ for $z = 0$ since the conductor is grounded. Thus $V(x,y,0) = \frac{1}{4\pi\epsilon_0\sqrt{x^2+y^2+d^2}}(q + q') = 0$, which gives $q' = -q$.

2. The force $\boldsymbol{F}_q$ on $q$ is

$$\boldsymbol{F}_q = q\boldsymbol{E}(0,0,d) = -q\nabla V\Big|_{(x,y,z)=(0,0,d)} \tag{28}$$

$$= -\frac{q^2}{4\pi\epsilon_0}\nabla\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]\Bigg|_{(x,y,z)=(0,0,d)}. \tag{29}$$

Here, the first term on the RHS is due to the field of the charge $q$ itself. This "self-force" is zero,[2] leaving only the contribution from the second term due to the image charge $-q$:

$$\begin{aligned} \boldsymbol{F}_q &= \frac{q^2}{4\pi\epsilon_0}\left(-\frac{1}{2}\right)(x^2+y^2+(z+d)^2)^{-3/2}(2x\hat{\boldsymbol{x}}+2y\hat{\boldsymbol{y}}+2(z+d)\hat{\boldsymbol{z}})|\Big|_{(x,y,z)=(0,0,d)} && (30)\\ &= -\frac{q^2}{4\pi\epsilon_0}\frac{2d\hat{\boldsymbol{z}}}{(2d)^3} = -\frac{q^2}{4\pi\epsilon_0(2d)^2}\hat{\boldsymbol{z}}. && (31)\end{aligned}$$

This result could have been found more easily by noting that $\boldsymbol{F}_q$ is given by force between $q$ and the image charge $-q$, which is of Coulomb form. The direction of the force is down, which makes sense, since physically the force is due to surface charge on the conductor which arises due to attraction to $q$ and thus has the opposite sign of $q$.

**(b)** 1. The surface charge density is (obtained e.g. by adapting Eq. (4) in the formula set, or by deriving it from Gauss's law)

$$\sigma(x,y) = -\epsilon_0\left[\frac{\partial V_{\rm above}}{\partial z} - \frac{\partial V_{\rm below}}{\partial z}\right]\Bigg|_{z=0}. \qquad (32)$$

Here $V_{\rm above}$ is the $V(x,y,z)$ that we found in (a), while $V_{\rm below}$ is the potential in the conductor, which is 0. Thus the second term on the RHS vanishes, leaving

$$\begin{aligned} \sigma(x,y) &= -\epsilon_0\frac{q}{4\pi\epsilon_0}\left(-\frac{1}{2}\right)\left[\frac{2(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{2(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right]\Bigg|_{z=0} && (33)\\ &= -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}. && (34)\end{aligned}$$

Note that this expression for the surface charge density $\sigma(x,y)$ has the following properties, all of which are as expected: (i) It has the opposite sign of $q$, (ii) it is radially symmetric in the $xy$ plane, i.e. its dependence on $x$ and $y$ can be expressed entirely in terms of the radial coordinate $s=\sqrt{x^2+y^2}$ in the $xy$ plane, (iii) it has dimensions charge/(distance squared), i.e. charge/area.

2. Let the total surface charge density be $Q$. The easiest way to find $Q$ is by considering the monopole term in the multipole expansion of $V(x,y,z)$ for large $r=\sqrt{x^2+y^2+z^2}$ (with $z>0$ so we are above the conductor). The monopole term $Q_{\rm tot}/4\pi\epsilon_0 r$ vanishes, since from the expression for $V(x,y,z)$, the total charge $Q_{\rm tot}=q+(-q)=0$. In the *physical* system (by which I mean the original system consisting of the charge $q$ and the conductor, in contrast to the fictitious system involving the image charge), any charge in addition to the point charge $q$ must be located on the surface of the conductor. Thus we must have $Q_{\rm tot}=q+Q$, which gives $Q=-q$.

Alternatively, $Q$ may be found by integrating $\sigma(x,y)$ over the $xy$ plane:

$$Q = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\sigma(x,y) = -\frac{qd}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\frac{1}{(x^2+y^2+d^2)^{3/2}}. \qquad (35)$$

Because of the radial symmetry of the integrand, we switch to polar coordinates $s$ and $\phi$. This gives

$$\begin{aligned} Q &= -\frac{qd}{2\pi}\int_0^{2\pi}d\phi\int_0^{\infty}ds\,s\,\frac{1}{(s^2+d^2)^{3/2}} \overset{u\equiv s^2+d^2}{=} -\frac{qd}{2\pi}\cdot 2\pi\cdot\frac{1}{2}\int_{d^2}^{\infty}du\,u^{-3/2} && (36)\\ &= -\frac{qd}{2}(-2)u^{-1/2}\Bigg|_{d^2}^{\infty} = qd(0-(d^2)^{-1/2}) = -q. && (37)\end{aligned}$$

[2] Some additional remarks on the self-force (not needed for solving the exam problem): Attempting to evaluate it directly fails due to the vanishing $(z-d)$. But one can use a more indirect way, by averaging the self-field over a sphere of radius $R$ centered on the charge, and then decreasing $R$ towards the radius of the particle (which for a true point particle is 0, which can lead to subtleties on its own). Since the self-field is of Coulomb form, its magnitude is the same at all points on the sphere, whereas its direction is radial. Thus the average of the self-field over the sphere vanishes, giving zero self-force. Interestingly, the self-force wouldn't vanish for an accelerated particle. This is related to the so-called *radiation reaction*, which we didn't have time to cover in this year's course.

3. On the surface of the conductor, an infinitesimal area element $da = dxdy$ that has charge $\sigma(x,y)da$ and is located at $\boldsymbol{r} = (x,y,0)$, acts on the charge $q$ at $\boldsymbol{r}_q = (0,0,d)$ with a force $d\boldsymbol{F}_q$ given by Coulomb's law:

$$d\boldsymbol{F}_q = \frac{q\sigma(x,y)dxdy}{4\pi\epsilon_0 R^2}\hat{\boldsymbol{R}} = \frac{q\sigma(x,y)dxdy}{4\pi\epsilon_0 R^3}\boldsymbol{R} \tag{38}$$

where I have here defined $\boldsymbol{R} \equiv \boldsymbol{r}_q - \boldsymbol{r} = -x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}} + d\hat{\boldsymbol{z}}$, $R = |\boldsymbol{R}| = \sqrt{x^2+y^2+d^2}$, and $\hat{\boldsymbol{R}} = \boldsymbol{R}/R$. The total force is found by integrating over the whole $xy$ plane:

$$\boldsymbol{F}_q = \int d\boldsymbol{F}_q = \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\frac{\sigma(x,y)}{(x^2+y^2+d^2)^{3/2}}(-x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}} + d\hat{\boldsymbol{z}}). \tag{39}$$

Since $\sigma(x,y)$ is an even function of $x$ ($y$), the term proportional to $\hat{\boldsymbol{x}}$ ($\hat{\boldsymbol{y}}$) involves an integrand that is odd in $x$ ($y$), so its integral vanishes. Therefore only the term proportional to $\hat{\boldsymbol{z}}$ survives, giving

$$\boldsymbol{F}_q = \int d\boldsymbol{F}_q = \hat{\boldsymbol{z}}\frac{qd}{4\pi\epsilon_0}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\frac{\sigma(x,y)}{(x^2+y^2+d^2)^{3/2}}. \tag{40}$$

(c) In this equation, $\Omega$ is an arbitrary volume and $a$ is the surface of $\Omega$.

- The first term: $\boldsymbol{F}$ is the total electromagnetic force (i.e. Lorentz force) on all charges inside $\Omega$.
- The second term: This is a surface integral over $a$, where $\overleftrightarrow{\boldsymbol{T}}$ is the Maxwell stress tensor with components $T_{ij}$, with the interpretation that $-\overleftrightarrow{\boldsymbol{T}}$ is the momentum flux density (momentum current density); more specifically, $-T_{ij}$ is the momentum in the $i$ direction crossing a surface oriented in the $j$ direction, per unit area, per unit time. The second term can therefore be interpreted as momentum current into $\Omega$. Alternatively, $\overleftrightarrow{\boldsymbol{T}}$ can be given a force-per-unit-area interpretation, with the diagonal components ($i = j$) being pressures and the off-diagonal components ($i \neq j$) being shears.
- The third term: Here $\boldsymbol{S}/c^2$ is the momentum density $g_{\text{EM}}$ of the electromagnetic field, where $\boldsymbol{S}$ is the Poynting vector. The volume integral thus gives the total momentum $\boldsymbol{p}_{\text{EM}}$ stored in the electromagnetic fields inside $\Omega$. The third term thus subtracts the time rate of change of $\boldsymbol{p}_{\text{EM}}$.

(The equation is closely connected to conservation of momentum. With $\boldsymbol{F} = \frac{d\boldsymbol{p}_{\text{mech}}}{dt}$, where $\boldsymbol{p}_{\text{mech}}$ is the total (mechanical) momentum of the charges inside $\Omega$, the equation expresses that the time rate of change of the total momentum $\boldsymbol{p}_{\text{mech}} + \boldsymbol{p}_{\text{EM}}$ in $\Omega$ equals the momentum current flowing into $\Omega$.)

**(d)** In this problem the third term in the equation disappears. This can be argued either because $\boldsymbol{B} = 0$ in this problem, so $\boldsymbol{S} = 0$, or because the problem is static, so $d/dt$ gives 0. We are thus left with

$$\boldsymbol{F}_q = \oint_a \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} \tag{41}$$

where $a$ is taken as the surface of the "upper half-sphere" of radius $R$ ($R \to \infty$), with the center of the sphere infinitesimally above $(0,0,0)$, so that the only charge inside the volume $\Omega$ enclosed by $a$ is the charge $q$. The surface area $a$ consists of the curved part of the half-sphere (the "northern hemisphere") and the "equatorial disk" in the $xy$ plane. Since static fields fall off at least as fast as $1/R^2$, and since $T_{ij}$ is quadratic in the fields, $\overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a}$ will fall off at least as fast as $(1/R^2)^2R^2 = 1/R^2$ on the curved part of the surface, which will thus not contribute for $R \to \infty$. Thus we are left with the contribution from the equatorial disk, which in the limit $R \to \infty$ becomes the whole $xy$ plane. Since $d\boldsymbol{a} = \hat{\boldsymbol{n}}\,da$ should point out of $\Omega$, the unit vector $\boldsymbol{n} = -\hat{\boldsymbol{z}}$. Thus (using Einstein's summation convention)

$$\begin{aligned}\overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} &= (T_{ij}\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j)\cdot(-da\hat{\boldsymbol{z}}) = -da\,T_{ij}\hat{\boldsymbol{e}}_i(\hat{\boldsymbol{e}}_j\cdot\hat{\boldsymbol{z}}) = -da\,T_{ij}\hat{\boldsymbol{e}}_i\delta_{jz} = -da\,T_{iz}\hat{\boldsymbol{e}}_i \\ &= -da(T_{xz}\hat{\boldsymbol{x}} + T_{yz}\hat{\boldsymbol{y}} + T_{zz}\hat{\boldsymbol{z}}).\end{aligned} \tag{42}$$

Since $\boldsymbol{E}$ just outside a conductor has no components parallel to the conductor surface, $E_x = E_y = 0$, so $T_{zx} = \epsilon_0 E_z E_x = 0$ and $T_{zy} = \epsilon_0 E_z E_y = 0$. This leaves the contribution from

$$T_{zz} = \epsilon_0\left(E_z^2 - \frac{1}{2}E^2\right) = \frac{\epsilon_0}{2}E_z^2 = \frac{\epsilon_0}{2}\left(-\frac{qd}{2\pi\epsilon_0(x^2+y^2+d^2)^{3/2}}\right)^2 = \frac{q^2d^2}{8\pi^2\epsilon_0(x^2+y^2+d^2)^3} \tag{43}$$

where I used $E_z = \sigma(x,y)/\epsilon_0$. Thus

$$\boldsymbol{F}_q = -\hat{\boldsymbol{z}}\frac{q^2 d^2}{8\pi^2\epsilon_0}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \frac{1}{(x^2+y^2+d^2)^3}. \tag{44}$$

(Incidentally, we can be sure that this expression is correct if we notice that one arrives at the same expression by inserting (34) into (40.) Using polar coordinates, the integral becomes

$$2\pi\int_0^\infty ds\, s\,\frac{1}{(s^2+d^2)^3} \overset{u\equiv s^2+d^2}{=} 2\pi\cdot\frac{1}{2}\int_{d^2}^\infty du\, u^{-3} = \pi\left(-\frac{1}{2}\right)u^{-2}\Big|_{d^2}^{\infty} = -\frac{\pi}{2}(0-(d^2)^{-2}) = \frac{\pi}{2d^4}. \tag{45}$$

Inserting this into (44) we arrive at the same result as in 2(a)2.

**Problem 3**

**(a)**
- The factors $\delta(x)$ and $\delta(y)$ are zero for $x \neq 0$ and $y \neq 0$, respectively, consistent with the wire going along the $z$ axis. As $\boldsymbol{j}$ is a vector, it is thus directed along the $z$ axis, which explains the factor $\hat{\boldsymbol{z}}$.
- As the wire goes between the two the charges at $z = \pm d/2$, it should vanish for $z > d/2$ and $z < -d/2$. This is taken care of by the Heaviside step function $\Theta(d/2 - |z)$.
- By definition of a current density $\boldsymbol{j}$, the surface integral $\int_a \boldsymbol{j}\cdot d\boldsymbol{a}$ gives the current through the (oriented) surface $a$. Taking $a$ to be plane parallel to the $xy$ plane, for a value of $z$ between $-d/2$ and $d/2$, oriented in the $+\hat{z}$ direction (i.e. the normal vector $\hat{\boldsymbol{n}}$ in $d\boldsymbol{a} = \hat{\boldsymbol{n}}dxdy$ is $\hat{\boldsymbol{n}} = +\hat{\boldsymbol{z}}$), the integral gives the current in the upward direction:

$$\hat{\boldsymbol{z}}\frac{dq}{dt}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} dy\,\delta(x)\delta(y)\cdot\hat{\boldsymbol{n}}dx\,dy = (\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{n}})\frac{dq}{dt} = +\frac{dq}{dt}. \tag{46}$$

This agrees with the fact that the upper charge is $q$ and the lower charge is $-q$.

**(b)** Defining the "complexified" version of $q$ as $\tilde{q}$, with $q = \mathrm{Re}(\tilde{q})$ (and similarly for other quantities), we can take $\tilde{q}(t) = q_0 e^{-i\omega t}$. This gives

$$\begin{aligned}\tilde{\boldsymbol{j}}(\boldsymbol{r},t) &= \hat{\boldsymbol{z}}\frac{d\tilde{q}(t)}{dt}\delta(x)\delta(y)\Theta(d/2-|z|) && (47)\\ &= -i\omega\hat{\boldsymbol{z}}\tilde{q}(t)\delta(x)\delta(y)\Theta(d/2-|z|). && (48)\end{aligned}$$

Thus, with $t_{\text{ret}} = t - |\boldsymbol{r}-\boldsymbol{r}'|/c$ being the retarded time, the complexified vector potential in the Lorenz gauge is (cf. Eq. (7) in the set of specific formulas):

$$\begin{aligned}\tilde{\boldsymbol{A}}(\boldsymbol{r},t) &= \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\tilde{\boldsymbol{j}}(\boldsymbol{r}',t_{\text{ret}})}{|\boldsymbol{r}-\boldsymbol{r}'|} && (49)\\ &= -i\omega\hat{\boldsymbol{z}}\frac{\mu_0}{4\pi}\int d^3r'\,\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\tilde{q}(t_{\text{ret}})\delta(x')\delta(y')\Theta(d/2-|z'|) && (50)\\ &= -i\omega q_0\hat{\boldsymbol{z}}\frac{\mu_0}{4\pi}\int d^3r'\,\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\exp\left[-i\omega(t-|\boldsymbol{r}-\boldsymbol{r}'|/c)\right]\delta(x')\delta(y')\Theta(d/2-|z'|). && (51)\end{aligned}$$

Because of the Dirac delta functions $\delta(x')$ and $\delta(y')$, doing the integrals over $x'$ and $y'$ is easy. As these integrations enforce $x' = y' = 0$, they reduce $\boldsymbol{r}'$ to $z'\hat{\boldsymbol{z}}$. The remaining integration over $z'$ runs from $-\infty$ to $+\infty$, but because the Heaviside step function is zero for $|z| > d/2$ and 1 for $|z| < d/2$, we may write

$$\tilde{\boldsymbol{A}}(\boldsymbol{r},t) = -i\omega q_0\hat{\boldsymbol{z}}\frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} dz'\,\frac{1}{|\boldsymbol{r}-z'\hat{\boldsymbol{z}}|}\exp\left[-i\omega(t-|\boldsymbol{r}-z'\hat{\boldsymbol{z}}|/c)\right]. \tag{52}$$

Since $|z'| < d/2$ and $d \ll r$, it follows that $|\boldsymbol{r}-z'\hat{\boldsymbol{z}}| \approx r$. Making this approximation corresponds to using just the leading term in the Taylor series expansion of $|\boldsymbol{r}-z'\hat{\boldsymbol{z}}|$ around $z' = 0$. The integrand then becomes independent of $z'$, giving

$$\tilde{\boldsymbol{A}}(\boldsymbol{r},t) \approx -i\omega\frac{\mu_0 q_0 d}{4\pi}\frac{e^{-i\omega(t-r/c)}}{r}\,\hat{\boldsymbol{z}}. \tag{53}$$

Considering also the 1st order term in the Taylor expansion, we would get two types of corrections in the integrand in (52) (as only the order of magnitude of the corrections matter in the following, I don't discuss their detailed form, such as Taylor series coefficients.) First, the factor $1/r$ would be replaced by $\frac{1}{r}(1+O(d/r))$ (here $O$ means "of the order of"). Thus the correction term goes like $1/r^2$ and therefore wouldn't contribute to radiation, so we drop it. Second, the factor $e^{i\omega r/c}$ would be replaced by $e^{i\omega(r+O(d))/c} = e^{i\omega r/c}e^{iO(\omega d/c)}$. Since by assumption $d \ll c/\omega$, i.e. $\omega d/c \ll 1$, we may approximate $e^{iO(\omega d/c)} \approx 1 + O(\omega d/c)$. Although the addition to 1 here would give a contribution to the radiation fields, we drop it since it is a small correction (i.e. keeping just the term 1 gives the *leading* part of the radiation fields). Thus we are left with (53). Taking the real part gives

$$\boldsymbol{A}_{\text{rad}}(\boldsymbol{r},t) = -\frac{\omega\mu_0 q_0 d}{4\pi}\frac{\sin[\omega(t-r/c)]}{r}\hat{\boldsymbol{z}}. \tag{54}$$

**(c)** To find $\boldsymbol{B}_{\text{rad}}$ we start from $\boldsymbol{B}_{\text{rad}} = \nabla \times \boldsymbol{A}_{\text{rad}}$ and throw away any terms that decay faster than $1/r$. We may in principle calculate the curl in any coordinate system we choose (cartesian, spherical, and cylindrical being the relevant ones for us). For this calculation I will use cylindrical coordinates ($s$, $\phi$, $z$). We see that $\boldsymbol{A}_{\text{rad}}$ only has a $z$ component, which only depends on $s$ and $z$ (via $r = \sqrt{s^2+z^2}$). Thus

$$\boldsymbol{B}_{\text{rad}}(\boldsymbol{r},t) = -\frac{\partial A_{\text{rad,z}}}{\partial s}\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\phi}}\frac{\partial r}{\partial s}\frac{\partial A_{\text{rad,z}}}{\partial r} = \hat{\boldsymbol{\phi}}\left(\frac{s}{r}\right)\frac{\omega\mu_0 q_0 d}{4\pi}\frac{\partial}{\partial r}\frac{\sin[\omega(t-r/c)]}{r}. \tag{55}$$

Using the product rule for differentiation, we only get a contribution to radiation from differentiating the sine function with respect to $r$ (the differentiation of $1/r$ gives an additional factor of $1/r$). Also note that $s/r = \cos(\pi/2-\theta) = \sin\theta$. Thus

$$\boldsymbol{B}_{\text{rad}}(\boldsymbol{r},t) = -\frac{\omega^2\mu_0 q_0 d}{4\pi c}\sin\theta\frac{\cos[\omega(t-r/c)]}{r}\hat{\boldsymbol{\phi}}. \tag{56}$$

**(d)** $\boldsymbol{E}_{\text{rad}}$ can be conveniently found from the 4th Maxwell equation, the Ampere-Maxwell law (as done in the lectures). This law involves $\boldsymbol{j}$, but this is not a problem, since we want to find $\boldsymbol{E}_{\text{rad}}$ far away from the dipole, where $\boldsymbol{j} = 0$ anyway. To calculate $\nabla \times \boldsymbol{B}_{\text{rad}}$ from (56) it is natural to use spherical coordinates. This gives

$$\frac{\partial \boldsymbol{E}_{\text{rad}}}{\partial t} = c^2\nabla\times\boldsymbol{B}_{\text{rad}} = c^2\left[\hat{\boldsymbol{r}}\frac{1}{r\sin\theta}\frac{\partial(\sin\theta B_{\text{rad},\phi})}{\partial\theta} - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial(rB_{\text{rad},\phi})}{\partial r}\right]. \tag{57}$$

We drop the term proportional to $\hat{\boldsymbol{r}}$ because it falls off like $1/r^2$. Thus

$$\frac{\partial \boldsymbol{E}_{\text{rad}}}{\partial t} = \frac{\omega^3\mu_0 q_0 d}{4\pi}\sin\theta\frac{\sin[\omega(t-r/c)]}{r}\hat{\boldsymbol{\theta}} \tag{58}$$

Integrating this expression with respect to $t$ for fixed $\boldsymbol{r}$ gives[3]

$$\boldsymbol{E}_{\text{rad}}(\boldsymbol{r},t) = -\frac{\omega^2\mu_0 q_0 d}{4\pi}\sin\theta\frac{\cos[\omega(t-r/c)]}{r}\hat{\boldsymbol{\theta}}. \tag{59}$$

**(e)** Let me first use (56) and (59). It follows from these equations that

$$|\boldsymbol{B}_{\text{rad}}(\boldsymbol{r},t)| = \frac{\omega^2\mu_0 q_0 d}{4\pi c}\sin\theta\frac{|\cos[\omega(t-r/c)]|}{r}, \quad |\boldsymbol{E}_{\text{rad}}(\boldsymbol{r},t)| = \frac{\omega^2\mu_0 q_0 d}{4\pi}\sin\theta\frac{|\cos[\omega(t-r/c)]|}{r}, \tag{60}$$

so (i) $|\boldsymbol{B}_{\text{rad}}|/|\boldsymbol{E}_{\text{rad}}| = 1/c$. Furthermore, $\boldsymbol{B}_{\text{rad}} \parallel \hat{\boldsymbol{\phi}}$ and $\boldsymbol{E}_{\text{rad}} \parallel \hat{\boldsymbol{\theta}}$. Therefore (ii) $\boldsymbol{B}_{\text{rad}} \perp \boldsymbol{E}_{\text{rad}}$ since $\hat{\boldsymbol{\phi}} \perp \hat{\boldsymbol{\theta}}$, and (iii) $\boldsymbol{B}_{\text{rad}}$, $\boldsymbol{E}_{\text{rad}} \perp \hat{\boldsymbol{r}}$ since $\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}} \perp \hat{\boldsymbol{r}}$.

Alternatively, one can use the expressions for $\boldsymbol{E}_{\text{rad}}$ and $\boldsymbol{B}_{\text{rad}}$ given in the problem text. We will also use that for general vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, the cross product $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$. Since both

[3] In principle the integration also gives an integration constant, i.e. a $t$-independent quantity (which could depend on $\boldsymbol{r}$). This is set to 0 since the time dependence of the problem isn't consistent with a time-independent contribution to $\boldsymbol{E}$ (this could be argued more carefully but I won't do that here).

$\boldsymbol{E}_{\rm rad}$ and $\boldsymbol{B}_{\rm rad}$ are proportional to the cross product between $\hat{\boldsymbol{r}}$ and another vector ($\ddot{\boldsymbol{p}}$ for $\boldsymbol{B}_{\rm rad}$ and $\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}$ for $\boldsymbol{E}_{\rm rad}$), it follows that (iii) $\boldsymbol{B}_{\rm rad}, \boldsymbol{E}_{\rm rad} \perp \hat{\boldsymbol{r}}$. Also, we can see from the given expressions that

$$\boldsymbol{E}_{\rm rad} = -c\hat{\boldsymbol{r}} \times \boldsymbol{B}_{\rm rad}, \tag{61}$$

so it similarly follows that (ii) $\boldsymbol{E}_{\rm rad} \perp \boldsymbol{B}_{\rm rad}$, and furthermore that (iii) $|\boldsymbol{E}_{\rm rad}| = c|\hat{\boldsymbol{r}}||\boldsymbol{B}_{\rm rad}|\sin(\pi/2) = c|\boldsymbol{B}_{\rm rad}|$.

(f) 1. We introduce a spherical surface $a$ defined by $r$ = constant ($\to \infty$). The time-averaged radiated power is

$$\langle P\rangle = \oint_a \langle \boldsymbol{S}\rangle \cdot d\boldsymbol{a}, \tag{62}$$

Inserting $d\boldsymbol{a} = \hat{\boldsymbol{r}}da = \hat{\boldsymbol{r}}r^2 d\Omega$ gives

$$\langle P\rangle = \oint_a \langle \boldsymbol{S}\rangle \cdot \hat{\boldsymbol{r}}r^2 d\Omega \equiv \int \langle\frac{dP}{d\Omega}\rangle d\Omega, \tag{63}$$

which gives

$$\langle\frac{dP}{d\Omega}\rangle = \langle \boldsymbol{S}\rangle \cdot \hat{\boldsymbol{r}}r^2. \tag{64}$$

In the following I will calculate the Poynting vector $\boldsymbol{S} = (1/\mu_0)(\boldsymbol{E}_{\rm rad} \times \boldsymbol{B}_{\rm rad})$ from the expressions for $\boldsymbol{E}_{\rm rad}$ and $\boldsymbol{B}_{\rm rad}$ given in the problem text (the alternative calculation of $\boldsymbol{S}$ directly from (56) and (59) is straightforward). Using (61) and vector identity (2) in the general formula set gives

$$\boldsymbol{S} = \frac{c}{\mu_0}\boldsymbol{B}_{\rm rad} \times (\hat{\boldsymbol{r}} \times \boldsymbol{B}_{\rm rad}) = \frac{c}{\mu_0}\left[\hat{\boldsymbol{r}}(\boldsymbol{B}_{\rm rad}\cdot\boldsymbol{B}_{\rm rad}) - \boldsymbol{B}_{\rm rad}\underbrace{(\boldsymbol{B}_{\rm rad}\cdot\hat{\boldsymbol{r}})}_{=0}\right] = \frac{c}{\mu_0}B_{\rm rad}^2\hat{\boldsymbol{r}}. \tag{65}$$

The electric dipole moment is

$$\boldsymbol{p}(t) = q(t)\frac{d}{2}\hat{\boldsymbol{z}} + (-q(t))(-\frac{d}{2}\hat{\boldsymbol{z}}) = q(t)d\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}q_0 d\cos\omega t = \hat{\boldsymbol{z}}p_0\cos\omega t \tag{66}$$

with $p_0 \equiv q_0 d$. Thus

$$\boldsymbol{B}_{\rm rad} = -\frac{\mu_0}{4\pi rc}\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}(t - r/c) = -\frac{\mu_0}{4\pi rc}(-\omega^2)p_0\cos[\omega(t-r/c)](\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}}) \tag{67}$$

$$= -\frac{\omega^2\mu_0 p_0}{4\pi c}\frac{\sin\theta}{r}\cos[\omega(t-r/c)]\hat{\boldsymbol{\phi}}, \tag{68}$$

where we used that $\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}} = \hat{\boldsymbol{\phi}}\sin\theta$. The result (68) is the same result as (56). Inserting it into (65) gives

$$\boldsymbol{S} = \frac{\omega^4\mu_0 p_0^2}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\cos^2[\omega(t-r/c)]\hat{\boldsymbol{r}}. \tag{69}$$

We take the time average using $\langle\cos^2[\omega(t-r/c)]\rangle = 1/2$. Then (64) gives

$$\langle\frac{dP}{d\Omega}\rangle = \frac{\omega^4\mu_0 p_0^2}{32\pi^2 c}\sin^2\theta. \tag{70}$$

We see that the radiation is maximal for $\theta = \pi/2$ and minimal for $\theta = 0$ and $\theta = \pi$.

2. Using $d\Omega = \sin\theta d\theta d\phi$ gives

$$\langle P\rangle = \int\langle\frac{dP}{d\Omega}\rangle d\Omega = \frac{\omega^4\mu_0 p_0^2}{32\pi^2 c}\int_0^{2\pi}d\phi\int_0^{\pi}\sin^2\theta\sin\theta d\theta. \tag{71}$$

Introducing $x = \cos\theta$, the $\theta$-integral can be rewritten as $\int_{-1}^{1}(1-x^2)dx = 2\left[x - \frac{1}{3}x^3\right]\Big|_0^1 = 2(1-1/3) = 4/3$, so

$$\langle P\rangle = \frac{\omega^4\mu_0 p_0^2}{12\pi c}. \tag{72}$$