TFY4240 Electromagnetic theory: Solution to exam, spring 2020

Problem 1

(a) It is convenient to take the z axis to coincide with the line and use cylindrical coordinates (s, ϕ, z). The electric field will only depend on the distance s to the line and will be parallel to the direction $\hat{\boldsymbol{s}}$. Thus $\boldsymbol{E} = E(s)\hat{\boldsymbol{s}}$ where $E(s)$ is to be determined. The high symmetry implies that Gauss's law will be useful. As the Gaussian surface we take a cylinder of radius s and length L whose axis coincides with the line. The form of $\boldsymbol{E}$ implies that the electric flux is nonzero only through the curved parts of the cylinder surface. Then Gauss's law gives

$$L \cdot 2\pi s \cdot E(s) = Q_{\text{inside}}/\epsilon_0 = \lambda L/\epsilon_0, \tag{1}$$

which gives

$$\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\boldsymbol{s}}. \tag{2}$$

Since $\boldsymbol{E}$ points in the $\hat{\boldsymbol{s}}$ direction, V will not depend on ϕ or z. To find V we may consider the line integral from a point P_0 with $s = s_0$ to a point P_1 with $s = s_1$. This gives

$$V(s_1) - V(s_0) = \int_{P_0}^{P_1} dV = -\int_{P_0}^{P_1} \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\lambda}{2\pi\epsilon_0}\int_{s_0}^{s_1} \frac{ds}{s} = -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{s_1}{s_0}. \tag{3}$$

Therefore (C is a constant)

$$V(s) = -\frac{\lambda}{2\pi\epsilon_0}\ln s + C = -\frac{\lambda}{2\pi\epsilon_0}\ln(x^2+y^2)^{1/2} + C = -\frac{\lambda}{4\pi\epsilon_0}\ln(x^2+y^2) + C. \tag{4}$$

(b) According to the method of images, the image charges must be placed outside the region where we wish to find the potential, i.e. they must be placed somewhere inside the region of the two conductors. By symmetry, the two image lines must intersect the x axis. Thus let the left (right) line have line charge density λ_L (λ_R) and x-coordinate b_L (b_R). Furthermore, there is no physical charge in the region outside the conductors. Thus we try the Ansatz

$$V(x,y) = -\frac{\lambda_L}{4\pi\epsilon_0}\ln[(x-b_L)^2+y^2] - \frac{\lambda_R}{4\pi\epsilon_0}\ln[(x-b_R)^2+y^2]. \tag{5}$$

The condition that the left conductor is grounded becomes $V(0,y) = 0$, i.e.

$$\lambda_L \ln[b_L^2+y^2] = -\lambda_R\ln[b_R^2+y^2] \tag{6}$$

which should hold for all y. For $y = 0$ this gives $\lambda_L/\lambda_R = -\ln(b_R^2)/\ln(b_L^2)$, while $|y| \to \infty$ gives $\lambda_L/\lambda_R = -1$. Thus $\lambda_L = -\lambda_R$ and $b_L = -b_R$ (having $b_L = b_R$ is also a solution, but it gives $V = 0$ everywhere and is therefore not acceptable for general V_{cyl}). In the following we write $\lambda_R \equiv \lambda$ and $b_R \equiv b$, giving

$$V(x,y) = -\frac{\lambda}{4\pi\epsilon_0}\ln\frac{(x-b)^2+y^2}{(x+b)^2+y^2}. \tag{7}$$

(c) The condition $V(x,y) = V_{\text{cyl}}$ for all (x,y) on the cylinder surface can be written

$$-\frac{\lambda}{4\pi\epsilon_0}\ln\frac{(x-b)^2+y^2}{(x+b)^2+y^2} = V_{\text{cyl}}. \tag{8}$$

This can be rewritten as

$$(x-b)^2+y^2 = W[(x+b)^2+y^2] \tag{9}$$

where we have defined the constant $W \equiv \exp(-4\pi\epsilon_0 V_{\text{cyl}}/\lambda)$. Multiplying out gives

$$(x^2+y^2)(1-W) - 2bx(1+W) + b^2(1-W) = 0. \tag{10}$$

The circular cross section of the cylinder surface implies that $(x-a)^2+y^2=R^2$, i.e. $x^2+y^2=R^2-a^2+2ax$. Inserting this into (10) gives

$$(R^2-a^2+2ax)(1-W)-2bx(1+W)+b^2(1-W)=0. \tag{11}$$

This must be true for any x on the cylinder surface, and thus the coefficient of x here must vanish, i.e.

$$2a(1-W)-2b(1+W)=0 \quad \Rightarrow \quad \frac{b}{a}=\frac{1-W}{1+W}. \tag{12}$$

The remaining part of (11) gives, after cancelling the common factor $(1-W)$, that $R^2-a^2+b^2=0$, i.e.

$$b=\sqrt{a^2-R^2}. \tag{13}$$

(From this it can be seen that $a-R<b<a+R$, so the right image line indeed lies inside the cylinder, as it should.) Next, let us temporarily write $4\pi\epsilon_0 V_{\rm cyl}/\lambda \equiv \alpha$, so $W=\exp(-\alpha)$. The expression for b/a in (12) can be rewritten as

$$\frac{b}{a} = \frac{1-W}{1+W}=\frac{1-e^{-\alpha}}{1+e^{-\alpha}}=\frac{e^{\alpha/2}-e^{-\alpha/2}}{e^{\alpha/2}+e^{-\alpha/2}}=\frac{\sinh(\alpha/2)}{\cosh(\alpha/2)}=\tanh(\alpha/2)=\tanh\left(\frac{2\pi\epsilon_0 V_{\rm cyl}}{\lambda}\right). \tag{14}$$

Solving for λ and inserting (13) gives

$$\lambda=\frac{2\pi\epsilon_0 V_{\rm cyl}}{\operatorname{arctanh}\left(\sqrt{1-(R/a)^2}\right)}. \tag{15}$$

(d) 1. Since (15) relates $V_{\rm cyl}$ and λ, we can obtain the relation between $V_{\rm cyl}$ and Λ by establishing a relation between λ and Λ. This can be done by applying Gauss's law $\oint \boldsymbol{E}\cdot d\boldsymbol{a}=Q_{\rm inside}/\epsilon_0$ to a cylinder-shaped Gaussian surface of some arbitrary length L, concentric with the physical cylinder and infinitesimally bigger than it in radius. On the RHS we have $Q_{\rm inside}=\Lambda L$. On the LHS we have to calculate the flux of $\boldsymbol{E}$ through the Gaussian surface. The flux through the two flat parts is zero, since $\boldsymbol{E}=0$ inside the cylinder. The flux through the curved part can be calculated from the potential V given by the sum of the contributions from the two image lines. The (net) electric flux from the left image line is zero since the same electric flux enters and leaves. The electric flux from the right image line is $2\pi RL\cdot\lambda/(2\pi\epsilon_0 R)=\lambda L/\epsilon_0$.[1] Thus Gauss's law gives $\lambda L/\epsilon_0=\Lambda L/\epsilon_0$ and thus $\lambda=\Lambda$. In other words, the line charge density Λ on the (real) cylinder equals the line charge density λ on the (fictitious) image line through the cylinder. It then follows from (15) that

$$V_{\rm cyl}=\frac{\Lambda}{2\pi\epsilon_0}\operatorname{arctanh}\left(\sqrt{1-(R/a)^2}\right). \tag{16}$$

2. For a positively charged cylinder (note that all the charge must be on the surface since the cylinder is a conductor), negative charge will be drawn to the surface of the left conductor. This will polarize the positive surface charge on the cylinder by shifting the surface charge distribution towards the left. Thus the surface charge density σ is expected to have a maximum at $\gamma=\pi$ and a minimum at $\gamma=0$ and 2π, with an angular average $\bar{\sigma}=\Lambda/(2\pi R)$. (This intuitive picture is confirmed by an exact calculation which gives $\sigma(\gamma)=\Lambda b/[2\pi R(a+R\cos\gamma)]$.)

Problem 2

(a) 1. I will limit the discussion here to the 12 interface, since the arguments are the same for the 23 interface. The boundary conditions that don't involve charges or currents are

$$\boldsymbol{E}_{1,\parallel}=\boldsymbol{E}_{2,\parallel} \quad \text{and} \quad B_{1,\perp}=B_{2,\perp}. \tag{17}$$

[1] Alternatively, we could work out these contributions by noting that both image lines give zero electric flux through the flat parts, so we may add the flat parts to get a *closed* surface, and then the electric flux is given by $Q_{\rm inside}/\epsilon_0$, where $Q_{\rm inside}$ equals 0 for the left image line and λL for the right image line.

In this problem the components parallel to the interface are the ϕ and z components, and the component perpendicular to the interface is the s component. Thus the boundary conditions can be rewritten as

$$E_{1,\phi} = E_{2,\phi}, \quad E_{1,z} = E_{2,z}, \quad B_{1,s} = B_{2,s}. \tag{18}$$

From the form of the EM wave in region 2, it follows that all region-2 components here are zero. So are all the region-1 components since $\boldsymbol{E}_1 = \boldsymbol{B}_1 = 0$. Thus the boundary conditions are satisfied.

2. The fields should satisfy the Maxwell equations in matter for region 2. Using $\rho_f = \boldsymbol{j}_f = 0$, $\boldsymbol{D} = \epsilon\boldsymbol{E}$, and $\boldsymbol{B} = \mu\boldsymbol{H}$, the Maxwell equations can be written

$$\nabla \cdot \boldsymbol{E} = 0, \tag{19}$$
$$\nabla \cdot \boldsymbol{B} = 0, \tag{20}$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{21}$$
$$\nabla \times \boldsymbol{B} = \epsilon\mu\frac{\partial \boldsymbol{E}}{\partial t}. \tag{22}$$

We now look up expressions for the divergence and curl in cylindrical coordinates and use the given form of the fields. Eq. (19) gives

$$\frac{\partial}{\partial s}(s\tilde{E}_0(s)) = 0, \tag{23}$$

Eq. (20) gives $0 = 0$, and Eq. (21) gives

$$\frac{\partial \tilde{E}_s}{\partial z} = -\frac{\partial \tilde{B}_\phi}{\partial t} \quad \Rightarrow \quad k\tilde{E}_0(s) = \omega\tilde{B}_0(s). \tag{24}$$

For Eq. (22) the analysis is a little more involved, as both the s and z components contribute. The s component gives

$$-\frac{\partial \tilde{B}_\phi}{\partial z} = \epsilon\mu\frac{\partial \tilde{E}_s}{\partial t} \quad \Rightarrow \quad k\tilde{B}_0(s) = \epsilon\mu\omega\tilde{E}_0(s), \tag{25}$$

and the z component gives

$$\frac{\partial}{\partial s}(s\tilde{B}_0(s)) = 0. \tag{26}$$

Eqs. (23) and (26) show that both $\tilde{E}_0(s)$ and $\tilde{B}_0(s)$ go like $1/s$. Combining (24) and (25) gives

$$k\tilde{E}_0(s) = \omega\tilde{B}_0(s) = (\epsilon\mu\omega^2/k)\tilde{E}_0(s), \tag{27}$$

i.e.

$$\frac{\omega}{k} = \frac{1}{\sqrt{\epsilon\mu}} \equiv v \tag{28}$$

and

$$\frac{\tilde{B}_0(s)}{\tilde{E}_0(s)} = \frac{1}{v}. \tag{29}$$

In summary,

$$\tilde{E}_0(s) = \frac{\tilde{F}}{s}, \quad \tilde{B}_0(s) = \frac{\tilde{F}}{vs} \tag{30}$$

where $\tilde{F}$ is a complex amplitude.

(b) The charge density can be found from Gauss's law:

$$\tilde{\rho} = \epsilon_0\nabla \cdot \tilde{\boldsymbol{E}} = 0. \tag{31}$$

Alternatively, it can be found as the sum of free and bound charge:

$$\tilde{\rho} = \tilde{\rho}_f + \tilde{\rho}_b = \tilde{\rho}_b = -\nabla \cdot \tilde{\boldsymbol{P}} = -\epsilon_0\chi_e\nabla \cdot \tilde{\boldsymbol{E}} = 0. \tag{32}$$

The current density can be found from the Ampere-Maxwell law:

$$\tilde{\boldsymbol{j}} = \frac{1}{\mu_0}\nabla\times\tilde{\boldsymbol{B}} - \epsilon_0\frac{\partial\tilde{\boldsymbol{E}}}{\partial t} = i\omega\left(\epsilon_0 - \frac{1}{v^2\mu_0}\right)\frac{\tilde{F}}{s}\exp[i(kz-\omega t)]\hat{\boldsymbol{s}}. \quad (33)$$

Alternatively, the current density can be found as the sum of free current, polarization current and bound (magnetization) current:

$$\tilde{\boldsymbol{j}} = \tilde{\boldsymbol{j}}_f + \tilde{\boldsymbol{j}}_P + \tilde{\boldsymbol{j}}_b = \tilde{\boldsymbol{j}}_P + \tilde{\boldsymbol{j}}_b, \quad (34)$$

where the polarization current density is

$$\tilde{\boldsymbol{j}}_P = \frac{\partial\tilde{\boldsymbol{P}}}{\partial t} = \epsilon_0\chi_e\frac{\partial\tilde{\boldsymbol{E}}}{\partial t} = -i\omega(\epsilon-\epsilon_0)\frac{\tilde{F}}{s}\exp[i(kz-\omega t)]\hat{\boldsymbol{s}}, \quad (35)$$

and the bound (magnetization) current density is

$$\tilde{\boldsymbol{j}}_b = \nabla\times\tilde{\boldsymbol{M}} = \frac{\chi_m}{\mu}\nabla\times\tilde{\boldsymbol{B}} = -ik\frac{\chi_m}{\mu}\frac{\tilde{F}}{vs}\exp[i(kz-\omega t)]\hat{\boldsymbol{s}} = -i\frac{\omega}{v^2}\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)\frac{\tilde{F}}{s}\exp[i(kz-\omega t)]\hat{\boldsymbol{s}}. \quad (36)$$

Adding these two contributions again gives (33) for the total current density. Finally it can be noted that the continuity equation $\partial\rho/\partial t = -\nabla\cdot\boldsymbol{j}$ is satisfied ($0 = 0$). Also, ρ and $\boldsymbol{j}$ vanish for the special case of vacuum, as they should.

(c) Here σ is the surface charge density and $\boldsymbol{K}$ is the surface current density. The equation takes the form of a continuity equation, i.e. a local conservation law for charge, but in contrast to the familiar continuity equation in 3 dimensions, this is a 2-dimensional version that refers to charge and current in the interface only.

The equation expresses how the charge $dq = \sigma da$ on an infinitesimal area element with area da in the interface varies in time by current leaving and/or entering the element. First, note that since $\boldsymbol{E} = \boldsymbol{B} = 0$ in the perfect conductors, it follows from the Ampere-Maxwell law that $\boldsymbol{j} = 0$ there. Thus no current can flow between the perfect conductors and the interfaces. Furthermore, no current can flow between region 2 and the interfaces since region 2 is vacuum with $\boldsymbol{j} = 0$ (Eq. (33) for the vacuum case). Thus any change of dq must be due to current flow in the interface. This leads in a natural way to this 2-dimensional version of the charge continuity equation.

(d) The argument for zero current between the perfect conductors and the interfaces is unchanged. But now region 2 is not vacuum, which permits a current in region 2, as given by (33). Thus a contribution to dq may come from current parallel to the direction $\hat{\boldsymbol{s}}$ perpendicular to the interfaces. The current from region 2 into an infinitesimal area element in an interface is $-j_s da$ at the 12 interface and $+j_s da$ at the 23 interface. This gives a positive contribution to $\partial dq/\partial t = da\,\partial\sigma/\partial t$, so that the continuity equation is modified to

$$\frac{\partial\sigma}{\partial t} = -\nabla_{\text{2D}}\cdot\boldsymbol{K} \mp j_s \quad (37)$$

where the upper (lower) sign is for the 12 (23) interface.

The correctness of Eq. (37) can be verified by explicit calculations, by finding σ and $\boldsymbol{K}$ from the two remaining boundary conditions. First consider the 12 interface. The general boundary conditions are

$$E_{2,\perp} - E_{1,\perp} = \sigma/\epsilon_0 \quad \text{and} \quad \boldsymbol{B}_{2,\parallel} - \boldsymbol{B}_{1,\parallel} = \mu_0(\boldsymbol{K}\times\boldsymbol{n}), \quad (38)$$

where $\boldsymbol{n}$ is a unit vector perpendicular to the interface, pointing from region 1 to region 2, and the signs of the $\perp$-components are with respect to this direction. Thus $\hat{\boldsymbol{n}} = \hat{\boldsymbol{s}}$, which gives

$$\sigma = \epsilon_0 E_{2,s} \quad \text{and} \quad \boldsymbol{K} = \frac{1}{\mu_0}\hat{\boldsymbol{s}}\times\boldsymbol{B}_{2,\parallel} = \frac{1}{\mu_0}\hat{\boldsymbol{s}}\times B_{2,\phi}\hat{\boldsymbol{\phi}} = \frac{1}{\mu_0}B_{2,\phi}\hat{\boldsymbol{z}}. \quad (39)$$

Using the expressions for the fields gives

$$\tilde{\sigma} = \epsilon_0 \frac{\tilde{F}}{s_i} \exp[i(kz - \omega t)] \quad \Rightarrow \quad \frac{\partial \tilde{\sigma}}{\partial t} = -i\omega\epsilon_0 \frac{\tilde{F}}{s_i} \exp[i(kz - \omega t)], \tag{40}$$

$$\tilde{\boldsymbol{K}} = \frac{1}{\mu_0} \frac{\tilde{F}}{v s_i} \exp[i(kz - \omega t)]\hat{\boldsymbol{z}} \quad \Rightarrow \quad \nabla_{2\mathrm{D}} \cdot \tilde{\boldsymbol{K}} = \frac{\partial \tilde{K}_z}{\partial z} = i\omega \frac{1}{v^2\mu_0} \frac{\tilde{F}}{s_i} \exp[i(kz - \omega t)]. \tag{41}$$

Thus

$$\frac{\partial \tilde{\sigma}}{\partial t} + \nabla_{2\mathrm{D}} \cdot \tilde{\boldsymbol{K}} = -i\omega \left(\epsilon_0 - \frac{1}{v^2 \mu_0} \right) \frac{\tilde{F}}{s_i} \exp[i(kz - \omega t)], \tag{42}$$

which indeed equals $-j_s$ at $s = s_i$, as seen from (33). At the 13 interface ($s = s_o$), the analysis is similar with $1 \to 3$ and $\hat{\boldsymbol{n}} \to -\hat{\boldsymbol{s}}$, so both σ and $\boldsymbol{K}$ change sign compared to at the 12 interface, which agrees with the positive sign on the rhs of (37) in that case.

Problem 3

(a) The Lorenz gauge expression for $\boldsymbol{A}$ is

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\boldsymbol{j}(\boldsymbol{r}', t_r)}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{43}$$

where $t_r = t - |\boldsymbol{r} - \boldsymbol{r}'|/c$ is the retarded time. For $t > 0$ the problem text describes a surface current in the xy plane with surface current density $K\hat{\boldsymbol{x}}$. Therefore, with $\boldsymbol{r}' = (x', y', z')$,

$$\boldsymbol{j}(\boldsymbol{r}', t_r) = K\delta(z')\Theta(t_r)\hat{\boldsymbol{x}} \tag{44}$$

where Θ is the Heaviside step function. Furthermore, since the current does not depend on x' or y', $\boldsymbol{A}(\boldsymbol{r}, t)$ cannot depend on x or y, so we may evaluate the integral for $x = y = 0$. Thus $|\boldsymbol{r} - \boldsymbol{r}'| = \sqrt{z^2 + x'^2 + y'^2}$, so

$$\boldsymbol{A}(\boldsymbol{r}, t) = \hat{\boldsymbol{x}} \frac{\mu_0 K}{4\pi} \int dx' \int dy' \frac{\Theta(t - \sqrt{z^2 + x'^2 + y'^2}/c)}{\sqrt{z^2 + x'^2 + y'^2}}. \tag{45}$$

The double integral here is best evaluated by going to polar coordinates. The integrand is independent of the angle ϕ' whose integral therefore just gives a factor 2π. This leaves the dependence on the radial coordinate $s' = \sqrt{x'^2 + y'^2}$, giving

$$\boldsymbol{A}(\boldsymbol{r}, t) = \hat{\boldsymbol{x}} \frac{\mu_0 K}{4\pi} \cdot 2\pi \int_0^\infty ds'\, s' \frac{\Theta(t - \sqrt{z^2 + s'^2}/c)}{\sqrt{z^2 + s'^2}}. \tag{46}$$

If $t < |z|/c$ the step function is zero for all s', giving $\boldsymbol{A} = 0$ then. For $t > |z|/c$ the step function is nonzero for $s' = 0$ to $s' = s'_{\max}$ where $ct = \sqrt{z^2 + s'^2_{\max}}$, giving $s'_{\max} = \sqrt{(ct)^2 - z^2}$. Thus

$$\boldsymbol{A}(\boldsymbol{r}, t) = \hat{\boldsymbol{x}} \frac{\mu_0 K}{2} \int_0^{s'_{\max}} ds' \frac{s'}{\sqrt{z^2 + s'^2}}. \tag{47}$$

Changing integration variable to $u = z^2 + s'^2$ gives $du = 2s'ds'$ and integration limits $u = z^2$ for $s' = 0$ and $u = z^2 + s'^2_{\max}$ for $s' = s'_{\max}$. Thus

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}, t) &= \hat{\boldsymbol{x}} \frac{\mu_0 K}{2} \frac{1}{2} \int_{z^2}^{z^2 + s'^2_{\max}} du\, u^{-1/2} = \hat{\boldsymbol{x}} \frac{\mu_0 K}{2} u^{1/2} \Big|_{z^2}^{z^2 + s'^2_{\max}} \\ &= \hat{\boldsymbol{x}} \frac{\mu_0 K}{2} (\sqrt{(ct)^2} - \sqrt{z^2}) = \hat{\boldsymbol{x}} \frac{\mu_0 K}{2} (ct - |z|). \end{aligned} \tag{48}$$

For a general t these results may be summarized as

$$\boldsymbol{A}(\boldsymbol{r}, t) = \hat{\boldsymbol{x}} \frac{\mu_0 K}{2} (ct - |z|)\Theta(t - |z|/c) \tag{49}$$

which is equivalent to the result stated in the problem text.

(b) Since there is no net charge anywhere, the scalar potential $V = 0$. Thus the electric field is

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} = -\hat{\boldsymbol{x}}\frac{\mu_0 K}{2}\frac{\partial}{\partial t}\left[(ct - |z|)\Theta(t - |z|/c)\right]. \tag{50}$$

Using the product rule for differentiation, the term obtained by differentiating the step function gives $(ct - |z|)\delta(t - |z|/c) = 0$. Therefore

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\hat{\boldsymbol{x}}\frac{\mu_0 K c}{2}\Theta(t - |z|/c). \tag{51}$$

The magnetic field is

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{\partial A_x}{\partial z}\hat{\boldsymbol{y}} \tag{52}$$

where we used the form of (49) in the last equality. Thus we must consider

$$\frac{\partial}{\partial z}\left[(ct - |z|)\Theta(t - |z|/c)\right] = -\Theta(t - |z|/c)\underbrace{\frac{\partial |z|}{\partial z}}_{\mathrm{sgn}(z)} + \underbrace{(ct - |z|)\delta(t - |z|/c)}_{0}\left(-\frac{1}{c}\right)\frac{\partial |z|}{\partial z}, \tag{53}$$

where $\mathrm{sgn}(z)$ is the sign of z. It follows that

$$\boldsymbol{B}(\boldsymbol{r}, t) = -\hat{\boldsymbol{y}}\frac{\mu_0 K}{2}\mathrm{sgn}(z)\Theta(t - |z|/c). \tag{54}$$

The fields are nonzero for $|z| < ct$. They therefore describe a wave with a wavefront at $|z| = ct$ which propagates outward from the xy plane (i.e. in the $+\hat{\boldsymbol{z}}$ direction for positive z and in the $-\hat{\boldsymbol{z}}$ direction for negative z) at speed c as time increases from $t = 0$. "Behind" this wavefront the fields are in phase with relative magnitude $|\boldsymbol{E}|/|\boldsymbol{B}| = c$, $\boldsymbol{E}$ and $\boldsymbol{B}$ are mutually perpendicular. The fields are plane waves since they are the same at every point in any plane perpendicular to the direction of propagation. On each side of the xy plane, these properties are similar to a harmonic plane EM wave propagating in the direction $\mathrm{sgn}(z)\hat{\boldsymbol{z}}$, except that the fields here have a very different dependence on time and space, being constant inside (behind) the wavefront and zero outside.

(c) Since there are no particles outside the xy plane, there are no mechanical contributions to the densities, only field contributions. The energy density is

$$u_{\mathrm{EM}} = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 = \frac{\mu_0 K^2}{4}\Theta(t - |z|/c). \tag{55}$$

The energy current density is the Poynting vector

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} = \frac{\mu_0 K^2 c}{4}\Theta(t - |z|/c)\mathrm{sgn}(z)\hat{\boldsymbol{z}}. \tag{56}$$

The momentum density is

$$\boldsymbol{g}_{\mathrm{EM}} = \frac{\boldsymbol{S}}{c^2} = \frac{\mu_0 K^2}{4c}\Theta(t - |z|/c)\mathrm{sgn}(z)\hat{\boldsymbol{z}}. \tag{57}$$

The momentum current density is $-\overleftrightarrow{\boldsymbol{T}}$, where $\overleftrightarrow{\boldsymbol{T}}$ is the Maxwell stress tensor. More precisely, $-T_{ij}$ is the momentum in the i direction crossing a surface oriented in the j direction, per unit area per unit time. We have found that the only nonzero field components are E_x and B_y. It follows that all the off-diagonal components of $\overleftrightarrow{T}$ vanish, i.e.

$$T_{ij} = 0 \quad \text{for } i \neq j. \tag{58}$$

Using the expressions for the fields, and $\epsilon_0 c^2 = 1/\mu_0$, the diagonal components are

$$T_{xx} = \epsilon_0(E_x^2 - \frac{1}{2}E^2) + \frac{1}{\mu_0}(B_x^2 - \frac{1}{2}B^2) = \frac{\epsilon_0}{2}E^2 - \frac{1}{2\mu_0}B^2 = 0, \quad (59)$$

$$T_{yy} = \epsilon_0(E_y^2 - \frac{1}{2}E^2) + \frac{1}{\mu_0}(B_y^2 - \frac{1}{2}B^2) = -\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 = 0, \quad (60)$$

$$T_{zz} = \epsilon_0(E_z^2 - \frac{1}{2}E^2) + \frac{1}{\mu_0}(B_z^2 - \frac{1}{2}B^2) = -\frac{\epsilon_0}{2}E^2 - \frac{1}{2\mu_0}B^2 = -\frac{\mu_0 K^2}{4}\Theta(t - |z|/c). \quad (61)$$

Thus the only nonzero component of the momentum current density is $-T_{zz}$.

(d) We have $\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}$. Earlier we calculated this by first doing the spatial integral in $\boldsymbol{A}$, and then taking the time derivative. If we instead first take the time derivative, we will arrive at what is Jefimenko's equation for $\boldsymbol{E}$ for this problem (where $\rho = \dot{\rho} = 0$):

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\frac{1}{4\pi\epsilon_0 c^2}\int d^3r' \frac{\frac{\partial}{\partial t}\boldsymbol{j}(\boldsymbol{r}', t_r)}{|\boldsymbol{r} - \boldsymbol{r}'|}. \quad (62)$$

Using (44) and the definition of t_r gives

$$\frac{\partial}{\partial t}\boldsymbol{j}(\boldsymbol{r}', t_r) = \frac{\partial}{\partial t}K\delta(z')\Theta(t_r)\hat{\boldsymbol{x}} = K\delta(z')\delta(t_r)\hat{\boldsymbol{x}}. \quad (63)$$

Let us first calculate $\boldsymbol{E}$ for $x = y = 0$:

$$\begin{aligned} \boldsymbol{E}(0,0,z,t) &= -\hat{\boldsymbol{x}}\frac{K}{4\pi\epsilon_0 c^2}\int dz'\,\delta(z')\int dx'\int dy'\,\frac{\delta(t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|} \\ &= -\hat{\boldsymbol{x}}\frac{K}{4\pi\epsilon_0 c^2}\int dx'\int dy'\,\frac{\delta(t - \sqrt{z^2 + x'^2 + y'^2}/c)}{\sqrt{z^2 + x'^2 + y'^2}} \\ &= -\hat{\boldsymbol{x}}\frac{K}{4\pi\epsilon_0 c^2}\cdot 2\pi\int_0^\infty ds'\,s'\,\frac{\delta(t - \sqrt{z^2 + s'^2}/c)}{\sqrt{z^2 + s'^2}}. \end{aligned} \quad (64)$$

The argument of the Dirac delta function is

$$t - \frac{1}{c}\sqrt{z^2 + s'^2} \equiv g(s'). \quad (65)$$

To calculate the integral, we use

$$\delta(g(s')) = \sum_i \frac{1}{|g'(s_i')|}\delta(s' - s_i') \quad (66)$$

where $\{s_i'\}$ are the zeroes of $g(s')$. The function $g(s')$ is zero at

$$s' = \sqrt{(ct)^2 - z^2} \equiv s_0' \quad (67)$$

provided that the argument of the square root is not negative (otherwise $g(s')$ has no zeroes, giving $\boldsymbol{E} = 0$). In this case the s' integral will have an integrand proportional to $\delta(s' - s_0')$, so there will only be a contribution from $s' = s_0'$, which is a circle with radius s_0' centered at the origin. By symmetry, for general values of x and y this will become a circle with radius s_0' centered at (x, y).

In conclusion, the only contribution to a nonzero $\boldsymbol{E}(x, y, z, t)$ comes from a circle in the xy plane, centered at (x, y) and with radius $\sqrt{(ct)^2 - z^2}$. (Thus the distance from the point (x, y, z) to any point on this circle is ct.)