TFY4240 Electromagnetic theory: Solution to exam, spring 2021

Problem 1

(a) For convenience I give the physical charge the label 0. The potential outside the conductor is

$$V = \sum_{i=0}^{3} V_i \quad \text{where} \quad V_i = \frac{q_i}{4\pi\epsilon_0\sqrt{(x-x_i)^2+(y-y_i)^2+z^2}}. \tag{1}$$

1. The potential V should be 0 everywhere on the conductor surface. By taking $q_1 = -q$, V_1 will cancel V_0 at the horizontal plane, but q_1 will also contribute $V_1 \neq 0$ at the vertical plane. By taking $q_3 = -q$, V_3 will cancel V_0 at the vertical plane, but q_3 will also contribute $V_3 \neq 0$ at the horizontal plane. The two unwanted contributions from q_1 and q_3 are cancelled by V_2 by taking $q_2 = +q$. Thus $V = 0$ is ensured by pairwise cancellations of the potentials V_i, as summarized here:

$$\text{Horizontal plane:} \quad V_0 + V_1 = 0 \quad \text{and} \quad V_2 + V_3 = 0, \tag{2}$$

$$\text{Vertical plane:} \quad V_0 + V_3 = 0 \quad \text{and} \quad V_1 + V_2 = 0. \tag{3}$$

2. The surface charge density is

$$\sigma = -\epsilon_0 \left(\partial_n V|_{\text{outside}} - \partial_n V|_{\text{inside}}\right). \tag{4}$$

where $\partial_n = \boldsymbol{n}\cdot\nabla$, with $\boldsymbol{n}$ the unit vector perpendicular to the conductor surface, pointing out of the conductor. The two terms refer to the derivatives just outside and just inside the conductor. The term labeled "inside" vanishes because V is constant in the conductor. For the horizontal plate, $\boldsymbol{n} = \hat{y}$, giving

$$\sigma = \sigma_h(x,z) = -\epsilon_0 \sum_{i=0}^{3} \partial_y V_i\big|_{y=0}. \tag{5}$$

Here

$$\partial_y V_i = \frac{q_i}{4\pi\epsilon_0}\cdot\left(-\frac{1}{2}\right)\frac{1}{[(x-x_i)^2+(y-y_i)^2+z^2]^{3/2}}\cdot 2(y-y_i)\cdot 1. \tag{6}$$

This gives

$$\begin{aligned}\sigma_h(x,z) &= -\frac{1}{4\pi}\sum_{i=0}^{3}\frac{q_i y_i}{[(x-x_i)^2+y_i^2+z^2]^{3/2}} \\ &= -\frac{qb}{2\pi}\left(\frac{1}{[(x-a)^2+b^2+z^2]^{3/2}} - \frac{1}{[(x+a)^2+b^2+z^2]^{3/2}}\right).\end{aligned} \tag{7}$$

The total charge on the horizontal plate is

$$Q_h = \int_0^\infty dx \int_{-\infty}^{\infty} dz\, \sigma_h(x,z). \tag{8}$$

The z-integral is

$$\int_{-\infty}^{\infty}\frac{dz}{[(x\pm a)^2+b^2+z^2]^{3/2}} = \frac{2}{(x\pm a)^2+b^2}. \tag{9}$$

Next, doing the x-integral gives

$$\begin{aligned}\int_0^\infty dx\,\frac{2}{[(x\pm a)^2+b^2]} &= \frac{2}{b}\arctan\left(\frac{x\pm a}{b}\right)\Bigg|_0^\infty \\ &= \frac{2}{b}\left(\arctan(\infty) - \arctan\left(\pm\frac{a}{b}\right)\right) = \frac{2}{b}\left(\frac{\pi}{2}\mp\arctan\left(\frac{a}{b}\right)\right).\end{aligned} \tag{10}$$

Thus

$$Q_h = -\frac{qb}{2\pi}\cdot\frac{2}{b}\left[\left(\frac{\pi}{2}+\arctan\left(\frac{a}{b}\right)\right) - \left(\frac{\pi}{2}-\arctan\left(\frac{a}{b}\right)\right)\right] = -\frac{2q}{\pi}\arctan\left(\frac{a}{b}\right). \tag{11}$$

3. By symmetry, Q_v can be found by interchanging a and b in Q_h. Thus, using also the given formula,

$$Q_h + Q_v = -\frac{2q}{\pi}\left[\arctan\left(\frac{a}{b}\right) + \arctan\left(\frac{b}{a}\right)\right] = -\frac{2q}{\pi}\cdot\frac{\pi}{2} = -q. \tag{12}$$

That this result makes sense can be argued as follows: The multipole expansion for V outside the conductor must be the same regardless of whether it is found from the fictitious system that gives (1) or from the physical system whose charges include q_0 and the surface charge density on the conductor. Thus in particular the total charge Q entering the monopole term can be written in two different ways:

$$Q = \sum_{i=0}^{3} q_i = q_0 + Q_h + Q_v. \tag{13}$$

Thus $Q_h + Q_v = \sum_{i=1}^{3} q_i = -q$, in agreement with (12). (More generally, this argument shows that the sum of the image charges equals the total charge on the conductor.)

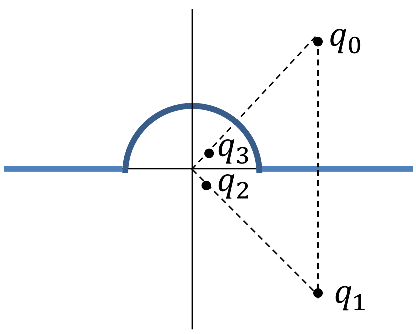
(b) Again V should be 0 everywhere on the conductor surface, which here consists of a horizontal part and a (hemi-)spherical part. This problem can be solved with a reasoning very similar to that used in 1(a)1. We follow the suggestion of making use of results from simpler problems. Two simpler (and previously encountered) problems in the category "a point charge outside a conductor" that have some overlap with this one have a conducting surface that is (i) an infinite horizontal plane and (ii) spherical.[1] A solution with 3 image charges q_1, q_2, q_3, placed as shown in the figure below, can be argued as follows:

The image charge $q_1 = -q$ will cancel V_0 at the horizontal part of the conductor surface but will also contribute $V_1 \neq 0$ at the spherical part. The image charge q_3 (with appropriate charge and position on the line passing through the origin and q_0) will cancel V_0 at the spherical part but will also contribute $V_3 \neq 0$ at the horizontal part. The two unwanted contributions from q_1 and q_3 can be cancelled by q_2 by taking q_2 to have charge $-q_3$ and be placed symmetrically to q_3 across the x-axis, as then $V = 0$ is ensured by pairwise cancellations of the potentials V_i as summarized here:

$$\text{Horizontal part:} \quad V_0 + V_1 = 0 \quad \text{and} \quad V_2 + V_3 = 0, \tag{14}$$

$$\text{Spherical part:} \quad V_0 + V_3 = 0 \quad \text{and} \quad V_1 + V_2 = 0. \tag{15}$$

It only remains to determine the charge and position of q_1. These are easily found by looking up the results for the charge and position of the single image charge in the simpler problem[1] and translating these to our situation. (It will involve the radius R and the distance $\sqrt{a^2+b^2}$ from the origin to q_0.) Thus V is given by Eq. (1), with the charges q_i and their positions $(x_i, y_i, 0)$ listed in the table below.



i	q_i	x_i	y_i
0	q	a	b
1	$-q$	a	$-b$
2	$\frac{qR}{\sqrt{a^2+b^2}}$	$\frac{aR^2}{a^2+b^2}$	$-\frac{bR^2}{a^2+b^2}$
3	$-\frac{qR}{\sqrt{a^2+b^2}}$	$\frac{aR^2}{a^2+b^2}$	$\frac{bR^2}{a^2+b^2}$

(c) The electric multipole expansion for V is $V(\boldsymbol{r}) = V_{\text{monopole}}(r) + V_{\text{dipole}}(\boldsymbol{r}) + V_{\text{quadrupole}}(\boldsymbol{r}) + \ldots$ The monopole moment is the total charge $Q = \sum_{i=0}^{3} q_i = 0$, so the monopole term $V_{\text{monopole}}(r) = Q/(4\pi\epsilon_0 r)$ vanishes. However, the dipole moment is nonzero:

$$\boldsymbol{p} = \sum_{i=0}^{3} q_i \boldsymbol{r}_i = 2qb\left(1 - \frac{R^3}{[a^2+b^2]^{3/2}}\right)\hat{y}. \tag{16}$$

[1]See Problem set 2 for the problem with a point charge outside a spherical conductor.

Thus from large distances the system looks like a dipole:

$$V(\boldsymbol{r}) \approx V_{\text{dipole}}(\boldsymbol{r}) = \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{4\pi\epsilon_0 r^2} = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\epsilon_0 r^3} = \frac{qby}{2\pi\epsilon_0[x^2+y^2+z^2]^{3/2}}\left(1 - \frac{R^3}{[a^2+b^2]^{3/2}}\right). \tag{17}$$

Problem 2

(a) See p. 9-11 in the lecture notes "Frequency-dependent response of materials".
(b) 1. The intensity of a wave $w = I, R, T$ is defined as the magnitude of the time-average of the associated Poynting vector, $I_w = |\langle \boldsymbol{S}_w \rangle|$. Using the formula

$$\langle CD \rangle = \frac{1}{2}\text{Re}(\tilde{C}^*\tilde{D}) \tag{18}$$

we find

$$\begin{aligned} \langle \boldsymbol{S}_I \rangle &= \frac{1}{2\mu_0}\text{Re}(\tilde{\boldsymbol{E}}_I^* \times \tilde{\boldsymbol{B}}_I) = \frac{1}{2\mu_0}\text{Re}(\tilde{E}_{0I}^* e^{-i(k_I z-\omega t)}\frac{1}{c}\tilde{E}_{0I}e^{i(k_I z-\omega t)}\hat{\boldsymbol{x}}\times\hat{\boldsymbol{y}}) \\ &= \frac{1}{2\mu_0 c}|\tilde{E}_{0I}|^2\,\hat{\boldsymbol{z}} = \frac{1}{2}\epsilon_0 c|\tilde{E}_{0I}|^2\,\hat{\boldsymbol{z}}. \end{aligned} \tag{19}$$

An almost identical calculation gives

$$\langle \boldsymbol{S}_R \rangle = -\frac{1}{2}\epsilon_0 c|\tilde{E}_{0R}|^2\,\hat{\boldsymbol{z}}. \tag{20}$$

Finally,

$$\begin{aligned} \langle \boldsymbol{S}_T \rangle &= \frac{1}{2\mu_0}\text{Re}(\tilde{E}_{0T}^* e^{-i(k_T^* z-\omega t)}\frac{\tilde{n}}{c}\tilde{E}_{0T}e^{i(k_T z-\omega t)}\hat{\boldsymbol{x}}\times\hat{\boldsymbol{y}}) \\ &= \frac{1}{2\mu_0 c}|\tilde{E}_{0T}|^2 e^{-2(\omega/c)n'' z}\text{Re}(\tilde{n})\hat{\boldsymbol{z}} = \frac{1}{2}\epsilon_0 c n'|\tilde{E}_{0T}|^2 e^{-2(\omega/c)n'' z}\hat{\boldsymbol{z}}. \end{aligned} \tag{21}$$

Thus

$$I_I = \frac{1}{2}\epsilon_0 c|\tilde{E}_{0I}|^2, \tag{22}$$

$$I_R = \frac{1}{2}\epsilon_0 c|\tilde{E}_{0R}|^2, \tag{23}$$

$$I_T = \frac{1}{2}\epsilon_0 c n'|\tilde{E}_{0T}|^2 e^{-2(\omega/c)n'' z}. \tag{24}$$

2.

$$R = \frac{I_R}{I_I} = \frac{|\tilde{E}_{0R}|^2}{|\tilde{E}_{0I}|^2} = \left|\frac{1-\tilde{n}}{1+\tilde{n}}\right|^2 = \frac{(n'-1)^2+(n'')^2}{(n'+1)^2+(n'')^2}, \tag{25}$$

$$T = \frac{I_T}{I_I} \overset{z=0}{=} n'\frac{|\tilde{E}_{0T}|^2}{|\tilde{E}_{0I}|^2} = \frac{4n'}{|1+\tilde{n}|^2} = \frac{4n'}{(n'+1)^2+(n'')^2}. \tag{26}$$

This gives

$$R + T = \frac{(n'-1)^2+(n'')^2+4n'}{(n'+1)^2+(n'')^2} = \frac{(n'+1)^2+(n'')^2}{(n'+1)^2+(n'')^2} = 1. \tag{27}$$

Thus the intensity of the incident wave is transferred to the reflected and transmitted waves. This is a manifestation/consequence of energy conservation.

(c) 1. There are several ways to prove this. I present three alternative proofs below. Also, in the Appendix I discuss some incorrect claims that were common in the answers to this question.

Proof 1. The time average of a periodic function $f(t)$ with period T is defined as

$$\langle f \rangle \equiv \frac{1}{T}\int_0^T dt\, f(t). \tag{28}$$

Thus

$$\left\langle \frac{\partial f}{\partial t} \right\rangle = \frac{1}{T}\int_0^T dt\, \frac{\partial f}{\partial t} = \frac{1}{T}(f(T) - f(0)) = 0. \tag{29}$$

This very short and simple proof is also the most general one, as f is a general periodic function here.

Proof 2. In the medium, the Poynting vector is $\boldsymbol{S} = \boldsymbol{S}_T$. Thus

$$\frac{\partial \boldsymbol{S}}{\partial t} = \frac{1}{\mu_0}\frac{\partial}{\partial t}(\boldsymbol{E}_T \times \boldsymbol{B}_T) = \frac{1}{\mu_0}\left(\frac{\partial \boldsymbol{E}_T}{\partial t} \times \boldsymbol{B}_T + \boldsymbol{E}_T \times \frac{\partial \boldsymbol{B}_T}{\partial t}\right). \tag{30}$$

Taking the time average gives

$$\left\langle \frac{\partial \boldsymbol{S}}{\partial t} \right\rangle = \frac{1}{\mu_0}\left(\left\langle \frac{\partial \boldsymbol{E}_T}{\partial t} \times \boldsymbol{B}_T \right\rangle + \left\langle \boldsymbol{E}_T \times \frac{\partial \boldsymbol{B}_T}{\partial t} \right\rangle\right). \tag{31}$$

Both fields have a simple harmonic time dependence with frequency ω. Their time derivatives are thus also simple harmonics. Thus we may use (18) to find the two time averages in (31). Using that

$$\widetilde{\frac{\partial \boldsymbol{E}_T}{\partial t}} = -i\omega \tilde{\boldsymbol{E}}_T, \quad \widetilde{\frac{\partial \boldsymbol{B}_T}{\partial t}} = -i\omega \tilde{\boldsymbol{B}}_T, \tag{32}$$

it follows that

$$\left\langle \frac{\partial \boldsymbol{S}}{\partial t} \right\rangle = \frac{1}{2\mu_0}\mathrm{Re}((-i\omega\tilde{\boldsymbol{E}}_T)^* \times \tilde{\boldsymbol{B}}_T + \tilde{\boldsymbol{E}}_T^* \times (-i\omega\tilde{\boldsymbol{B}}_T)) = \frac{\omega}{2\mu_0}\mathrm{Re}[(i-i)\tilde{\boldsymbol{E}}_T^* \times \tilde{\boldsymbol{B}}_T) = \frac{\omega}{2\mu_0}\mathrm{Re}(0) = 0. \tag{33}$$

Proof 3. This proof uses the physical fields, not the complex ones. First, we write the complex amplitude $\tilde{E}_{0T}$ and the complex refractive index in polar form by introducing their magnitudes and phase angles:

$$\tilde{E}_{0T} = |\tilde{E}_{0T}|e^{i\delta_{0T}}, \tag{34}$$

$$\tilde{n} = |\tilde{n}|e^{i\phi}. \tag{35}$$

This gives

$$\tilde{\boldsymbol{E}}_T = |\tilde{E}_{0T}|e^{i(\omega/c)n'z - \omega t + \delta_{0T})}e^{-(\omega/c)n''z}\,\hat{\boldsymbol{x}}, \tag{36}$$

$$\tilde{\boldsymbol{B}}_T = \frac{|\tilde{n}|}{c}|\tilde{E}_{0T}|e^{i(\omega/c)n'z - \omega t + \delta_{0T} + \phi)}e^{-(\omega/c)n''z}\,\hat{\boldsymbol{y}}. \tag{37}$$

Thus

$$\boldsymbol{E}_T = \mathrm{Re}(\tilde{\boldsymbol{E}}_T) = |\tilde{E}_{0T}|\cos((\omega/c)n'z - \omega t + \delta_{0T})e^{-(\omega/c)n''z}\,\hat{\boldsymbol{x}}, \tag{38}$$

$$\boldsymbol{B}_T = \mathrm{Re}(\tilde{\boldsymbol{B}}_T) = \frac{|\tilde{n}|}{c}|\tilde{E}_{0T}\cos((\omega/c)n'z - \omega t + \delta_{0T} + \phi)e^{-(\omega/c)n''z}\,\hat{\boldsymbol{y}}. \tag{39}$$

Thus

$$\boldsymbol{S}_T = \frac{1}{\mu_0}\boldsymbol{E}_T \times \boldsymbol{B}_T = \frac{|\tilde{n}|}{\mu_0 c}|\tilde{E}_{0T}|^2\cos((\omega/c)n'z - \omega t + \delta_{0T})\cos((\omega/c)n'z - \omega t + \delta_{0T} + \phi)e^{-2(\omega/c)n''z}\,\hat{\boldsymbol{z}}. \tag{40}$$

This looks complicated, but note that the only thing of importance here is the time dependence. Only the two cosine factors are time-dependent, and their time dependence is restricted to the $-\omega t$ terms, so everything else in each cosine argument can be lumped into a constant. Name the two constants γ and η, i.e. $\gamma = (\omega/c)n'z + \delta_{0T}$ and $\eta = (\omega/c)n'z + \delta_{0T} + \phi$. Thus the product of the two cosines is

$$\cos(\gamma - \omega t)\cos(\eta - \omega t). \tag{41}$$

Its time derivative is

$$\begin{aligned}\frac{d}{dt}[\cos(\gamma - \omega t)\cos(\eta - \omega t)] &= \omega[\sin(\gamma - \omega t)\cos(\eta - \omega t) + \cos(\gamma - \omega t)\sin(\eta - \omega t)] \\ &= \omega\sin(\gamma + \eta - 2\omega t). \end{aligned} \tag{42}$$

The function $\sin(\gamma + \eta - 2\omega t)$ is harmonic with period $T/2$, so its time average vanishes.

2. The total force on the charges and currents inside a volume Ω is

$$\boldsymbol{F} = \int_\Omega d^3r\, \boldsymbol{f} = \int_\Omega d^3r \left[\nabla\cdot\overleftrightarrow{T} - \frac{1}{c^2}\frac{\partial \boldsymbol{S}}{\partial t}\right] = \int_a d\boldsymbol{a}\cdot\overleftrightarrow{T} - \frac{1}{c^2}\int_\Omega d^3r\,\frac{\partial \boldsymbol{S}}{\partial t}. \tag{43}$$

In the last transition we rewrote as usual the volume integral of $\nabla\cdot\overleftrightarrow{T}$ as a surface integral by using a divergence theorem; in the resulting surface integral a is the closed surface that bounds the volume Ω. Taking the time average gives

$$\langle\boldsymbol{F}\rangle = \int_a d\boldsymbol{a}\cdot\langle\overleftrightarrow{T}\rangle - \frac{1}{c^2}\int_\Omega d^3r\left\langle\frac{\partial \boldsymbol{S}}{\partial t}\right\rangle. \tag{44}$$

We want the force on the medium; hence the volume Ω should be chosen to contain the medium in its interior (and no other charges or currents; but that is not a complication here since there are none outside the medium). We also want to make full use of the result proved in 2(c)1; hence this volume should surround the medium "tightly" so that we can use that $\left\langle\frac{\partial \boldsymbol{S}}{\partial t}\right\rangle = 0$ inside Ω. Thus with such a choice of Ω,

$$\langle\boldsymbol{F}\rangle = \int_a d\boldsymbol{a}\cdot\langle\overleftrightarrow{T}\rangle. \tag{45}$$

Since the medium is (semi-)infinite we analyze a finite volume Ω that in the infinite-volume limit includes the entire medium. Given the simple field dependence on the cartesian coordinates in this problem, i.e.

$$\boldsymbol{E} = E(z)\hat{\boldsymbol{x}}, \qquad \boldsymbol{B} = B(z)\hat{\boldsymbol{y}}, \tag{46}$$

we pick Ω to be a box (rectangular prism) with side lengths L_x, L_y, L_z. In the x direction we can take the box to go from $x = -L_x/2$ to $L_x/2$, in the y direction from $y = -L_y/2$ to $L_y/2$, while in the z direction it should go from $z = 0$ to L_z. Then the box will contain the entire medium in the limit $L_x, L_y, L_z \to \infty$.

Next we analyze the contributions to (45) from the six faces of the box. For this discussion, we will use that $d\boldsymbol{a} = da\,\hat{\boldsymbol{n}}$ where the unit vector $\hat{\boldsymbol{n}}$ should point out of the box.

- For the two faces with $x = \pm L_x/2$, note that for each point $(L_x/2, y, z)$ on the face with $x = +L_x/2$ with $\hat{\boldsymbol{n}} = +\hat{\boldsymbol{x}}$, there is an opposite point $(-L_x/2, y, z)$ with $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{x}}$. As $\overleftrightarrow{T}$ is identical at these points (because the field components are x-independent, cf. (46)), the contributions $d\boldsymbol{a}\cdot\overleftrightarrow{T}$ from such pairs of points cancel. Thus these two faces give zero net contribution.
- An exactly analogous argument holds for the two faces with $y = \pm L_y/2$: For each point $(x, L_y/2, z)$ on the face with $y = +L_y/2$ with $\hat{\boldsymbol{n}} = +\hat{\boldsymbol{y}}$, there is an opposite point $(x, -L_y/2, z)$ with $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{y}}$. As $\overleftrightarrow{T}$ is identical at these points (because the field components are y-independent, cf. (46)), the contributions $d\boldsymbol{a}\cdot\overleftrightarrow{T}$ from such pairs of points cancel. Thus these two faces give zero net contribution.

- For the face at $z = L_z$, the nonzero components of $\overleftrightarrow{T}$ will decay exponentially as $e^{-2(\omega/c)n'' L_z}$ (because the fields in the medium decay like $e^{-(\omega/c)n'' z}$), so the contribution goes to 0 as $L_z \to \infty$.

- For the face at $z = 0$, $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{z}}$, so

$$d\boldsymbol{a} \cdot \overleftrightarrow{T} = -da\, \hat{\boldsymbol{z}} \cdot T_{ij} \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j = -daT_{ij}(\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{x}}_i)\hat{\boldsymbol{x}}_j = -daT_{ij}\delta_{iz}\hat{\boldsymbol{x}}_j = -da(T_{zx}\hat{\boldsymbol{x}} + T_{zy}\hat{\boldsymbol{y}} + T_{zz}\hat{\boldsymbol{z}}). \quad (47)$$

  Since $E_z = B_z = 0$ it follows that $T_{zx} = T_{zy} = 0$. Thus

$$d\boldsymbol{a} \cdot \overleftrightarrow{T} = -da\, T_{zz}\hat{\boldsymbol{z}}. \quad (48)$$

In conclusion, only the face at $z = 0$, i.e. at the vacuum-medium interface, will contribute. For this face $da = dxdy$. Since the field components depend on neither x nor y (cf. (46)), the surface integral over the interface will become proportional to its area A. Thus the force will diverge as $A \to \infty$. But the force per unit area will be finite:

$$\frac{\langle \boldsymbol{F} \rangle}{A} = -\langle T_{zz} \rangle\, \hat{\boldsymbol{z}}.$$

It remains to evaluate $\langle T_{zz} \rangle$. We have

$$\begin{aligned} T_{zz} &= \epsilon_0(E_z^2 - \frac{1}{2}E^2) + \frac{1}{\mu_0}(B_z^2 - \frac{1}{2}B^2) = -\frac{\epsilon_0}{2}\boldsymbol{E}^2 - \frac{1}{2\mu_0}\boldsymbol{B}^2 \\ \Rightarrow \quad \langle T_{zz} \rangle &= -\frac{\epsilon_0}{2}\langle \boldsymbol{E}^2 \rangle - \frac{1}{2\mu_0}\langle \boldsymbol{B}^2 \rangle. \end{aligned} \quad (50)$$

Using (18), the time averages can be found as

$$\langle \boldsymbol{E} \cdot \boldsymbol{E} \rangle = \frac{1}{2}\mathrm{Re}(\tilde{\boldsymbol{E}}^* \cdot \tilde{\boldsymbol{E}}), \quad (51)$$

$$\langle \boldsymbol{B} \cdot \boldsymbol{B} \rangle = \frac{1}{2}\mathrm{Re}(\tilde{\boldsymbol{B}}^* \cdot \tilde{\boldsymbol{B}}). \quad (52)$$

Since the medium should be completely *inside* Ω, we evaluate T_{zz} *just outside* the interface, on the vacuum side (i.e. at $z = 0^-$), where $\boldsymbol{E} = \boldsymbol{E}_I + \boldsymbol{E}_R$ and $\boldsymbol{B} = \boldsymbol{B}_I + \boldsymbol{B}_R$. One finds

$$\begin{aligned} \tilde{\boldsymbol{E}}^* \cdot \tilde{\boldsymbol{E}} &= (\tilde{\boldsymbol{E}}_I^* + \tilde{\boldsymbol{E}}_R^*) \cdot (\tilde{\boldsymbol{E}}_I + \tilde{\boldsymbol{E}}_R) \\ &= \tilde{\boldsymbol{E}}_I^* \cdot \tilde{\boldsymbol{E}}_I + \tilde{\boldsymbol{E}}_R^* \cdot \tilde{\boldsymbol{E}}_R + \tilde{\boldsymbol{E}}_I^* \cdot \tilde{\boldsymbol{E}}_R + \tilde{\boldsymbol{E}}_R^* \cdot \tilde{\boldsymbol{E}}_I \\ &= |\tilde{E}_{0I}|^2 + |\tilde{E}_{0R}|^2 + \tilde{E}_{0I}^*\tilde{E}_{0R}e^{-2ik_I z} + \tilde{E}_{0R}^*\tilde{E}_{0I}e^{2ik_I z}, \end{aligned} \quad (53)$$

$$\begin{aligned} \tilde{\boldsymbol{B}}^* \cdot \tilde{\boldsymbol{B}} &= (\tilde{\boldsymbol{B}}_I^* + \tilde{\boldsymbol{B}}_R^*) \cdot (\tilde{\boldsymbol{B}}_I + \tilde{\boldsymbol{B}}_R) \\ &= \tilde{\boldsymbol{B}}_I^* \cdot \tilde{\boldsymbol{B}}_I + \tilde{\boldsymbol{B}}_R^* \cdot \tilde{\boldsymbol{B}}_R + \tilde{\boldsymbol{B}}_I^* \cdot \tilde{\boldsymbol{B}}_R + \tilde{\boldsymbol{B}}_R^* \cdot \tilde{\boldsymbol{B}}_I \\ &= \frac{1}{c^2}(|\tilde{E}_{0I}|^2 + |\tilde{E}_{0R}|^2 - \tilde{E}_{0I}^*\tilde{E}_{0R}e^{-2ik_I z} - \tilde{E}_{0R}^*\tilde{E}_{0I}e^{2ik_I z}). \end{aligned} \quad (54)$$

Note that the cross terms coupling the incident and reflected waves in (54) have the opposite sign from those in (53); this comes from $\boldsymbol{k}_R = -\boldsymbol{k}_I$ in the expression $\boldsymbol{B}_w = \frac{1}{c}\hat{\boldsymbol{k}}_w \times \boldsymbol{E}_w$ ($w = I, R$). As a consequence of this, and $1/\mu_0 = \epsilon_0 c^2$, the contribution from these cross terms cancel in $\langle T_{zz} \rangle$, leaving

$$\langle T_{zz} \rangle = -\frac{1}{2}\epsilon_0(|\tilde{E}_{0I}|^2 + |\tilde{E}_{0R}|^2). \quad (55)$$

Using (22) and (25) this can be expressed as $\langle T_{zz} \rangle = -\frac{I_I}{c}(1 + R)$ and therefore

$$\frac{\langle \boldsymbol{F} \rangle}{A} = \frac{I_I}{c}(1 + R)\, \hat{\boldsymbol{z}}. \quad (56)$$

3. The result (56) shows that the force points in the $\hat{\boldsymbol{z}}$ direction, which makes intuitive sense since the incident wave propagates in this direction. The magnitude of (56) is a pressure, usually called the radiation pressure:

$$\mathcal{P}_{\mathrm{rad}} = \frac{I_I}{c}(1 + R). \quad (57)$$

It also makes sense that this pressure should increase with the intensity I_I of the incident wave. We also see that it increases with R, with the two extreme cases being

$$R = 0: \qquad \mathcal{P}_{\text{rad}} = \frac{I_I}{c}, \tag{58}$$

$$R = 1: \qquad \mathcal{P}_{\text{rad}} = \frac{2I_I}{c}. \tag{59}$$

This is related to momentum conservation. When $R = 0$ there is no reflected wave, so the momentum of the incident wave is transferred to the medium. When $R = 1$ the momentum of the reflected wave has the same magnitude but opposite sign of the incident wave. Momentum conservation then implies that twice as much momentum is transferred to the medium.[2]

Problem 3

(a) 1. Here $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_q$, where $\boldsymbol{r}_q$ is the position of the particle, $\boldsymbol{v}$ is its velocity, and $\boldsymbol{a}$ its acceleration. A crucial fact is that due to the finite speed c of electromagnetic signals, the particle properties $\boldsymbol{r}_q$, $\boldsymbol{v}$, and $\boldsymbol{a}$ should not be evaluated at time t, but instead at the earlier so-called retarded time t_{ret} defined by the equation

$$t_{\text{ret}} = t - \frac{|\boldsymbol{r} - \boldsymbol{r}_q(t_{\text{ret}})|}{c}. \tag{60}$$

2. In order to evaluate the fields at $\boldsymbol{r} = (x, y, z)$ at time t it is necessary to solve Eq. (60) for the retarded time t_{ret}. For this problem it becomes

$$t_{\text{ret}} = t - \frac{\sqrt{x^2 + y^2 + (z - z_0\cos(\omega t_{\text{ret}}))^2}}{c}. \tag{61}$$

3. This question[3] can be answered for the general Eq. (60). The number of solutions for a given particle trajectory and given $(\boldsymbol{r}, t)$ is equal to the number of intersections of the worldline of the particle (obtained from $\boldsymbol{r}_q(t)$) with the past light cone of the spacetime point $(\boldsymbol{r}, t)$ in a spacetime diagram (because an electromagnetic signal sent from such an intersection point will pass through $(\boldsymbol{r}, t)$). The particle worldline is bound to intersect the past light cone. Furthermore, there cannot be more than one intersection, as this would imply that the particle would have had to move at a velocity exceeding c during at least some of the time between two such intersections, and this is impossible. Therefore there is one solution for t_{ret}.[4]

(b) We may neglect the "velocity field" that decays like $1/R^2$ in $\boldsymbol{E}$, as it will not contribute to radiation. This leaves the "acceleration field"

$$\boldsymbol{E}_{\text{acc}}(\boldsymbol{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{R}{(\boldsymbol{R}\cdot\boldsymbol{u})^3}\boldsymbol{R}\times(\boldsymbol{u}\times\boldsymbol{a}). \tag{62}$$

For general $(\boldsymbol{r}, t)$ this is not easy to evaluate, not least because we don't have an explicit expression for t_{ret} (cf. 3(a)2.) However, since the question is concerned only with the combined conditions

$$z_0/r \ll 1 \qquad \text{and} \qquad v/c \ll 1, \tag{63}$$

the ratios appearing here play the role of small expansion parameters, so we are justified in neglecting small non-leading contributions. This will simplify the expressions considerably. For example, one can argue as follows. First, from $\boldsymbol{v}(t_{\text{ret}}) = -\omega z_0 \sin(\omega t_{\text{ret}})$ we see that the maximum speed is ωz_0. Thus $v/c \ll 1$ implies $\omega z_0/c \ll 1$, which it will be useful to express as

$$\frac{\omega O(z_0)}{c} \ll 1, \tag{64}$$

[2] Discussed in the 15/3 lecture.

[3] Discussed in the 22/3 lecture.

[4] This is why we refer to *the* retarded time. Had there been more than one retarded time, each would have contributed to the fields, and we would have had to add the different contributions.

where $O(z_0)$ means z_0 multiplied by a dimensionless number of order 1. Next, note that $|\boldsymbol{r} - \boldsymbol{r}_q(t_{\rm ret})| = r + O(z_0)$. Thus

$$\omega t_{\rm ret} \;=\; \omega\left(t - \frac{|\boldsymbol{r} - \boldsymbol{r}_q(t_{\rm ret})|}{c}\right) = \omega\left(t - \frac{r + O(z_0)}{c}\right) = \omega(t - r/c) + \frac{\omega O(z_0)}{c}. \tag{65}$$

Thus the cosine factor in $z_q(t_{\rm ret}) = z_0\cos(\omega t_{\rm ret})$ becomes

$$\begin{aligned}\cos(\omega t_{\rm ret}) &= \cos\left[\omega(t - r/c) + \frac{\omega O(z_0)}{c}\right]\\ &= \cos[\omega(t - r/c)]\cos\left[\frac{\omega O(z_0)}{c}\right] - \sin\left[\omega(t - r/c)\right]\sin\left[\frac{\omega O(z_0)}{c}\right]\\ &\approx \cos[\omega(t - r/c)],\end{aligned} \tag{66}$$

where the last approximation could be made because (64) implies $\cos\left[\frac{\omega O(z_0)}{c}\right] \approx 1$ and $\sin\left[\frac{\omega O(z_0)}{c}\right] \approx \frac{\omega O(z_0)}{c} \ll 1$. Thus we are justified in making the approximation $t_{\rm ret} \approx t - r/c$ in $\omega t_{\rm ret}$. Therefore

$$\boldsymbol{a}(t_{\rm ret}) \approx \boldsymbol{a}(t - r/c) = -\omega^2 z_0 \cos[\omega(t - r/c)]\hat{\boldsymbol{z}}. \tag{67}$$

The first condition in (63) implies that in other places where $\boldsymbol{R}$, R and $\hat{\boldsymbol{R}}$ appear we can approximate it with $\boldsymbol{r}$, r and $\hat{\boldsymbol{r}}$, and the second condition implies that we can neglect terms of $O(v/c)$, so $\boldsymbol{u} \approx c\hat{\boldsymbol{r}}$. Inserting these approximations into (62) gives the radiation field

$$\boldsymbol{E}_{\rm rad}(\boldsymbol{r}, t) \approx \frac{q}{4\pi\epsilon_0 c^2 r}\,\hat{\boldsymbol{r}} \times [\hat{\boldsymbol{r}} \times \boldsymbol{a}(t - r/c)]. \tag{68}$$

To evaluate the two cross products, we start with the inner one, using in turn (67) and $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{z}} = -\sin\theta\,\hat{\boldsymbol{\varphi}}$, and then $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\varphi}} = -\hat{\boldsymbol{\theta}}$ for the outer one. As a result, Eq. (68) becomes

$$\boldsymbol{E}_{\rm rad}(\boldsymbol{r}, t) \approx -\frac{q z_0 \omega^2}{4\pi\epsilon_0 c^2 r}\cos[\omega(t - r/c)]\sin\theta\,\hat{\boldsymbol{\theta}}. \tag{69}$$

Now the magnetic field can be found from $(1/c)\hat{\boldsymbol{R}} \times \boldsymbol{E}_{\rm acc}(\boldsymbol{r}, t)$. Making the same approximations as for $\boldsymbol{E}$ gives $(1/c)\hat{\boldsymbol{r}} \times \boldsymbol{E}_{\rm rad}(\boldsymbol{r}, t)$. Finally, using $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ gives

$$\boldsymbol{B}_{\rm rad}(\boldsymbol{r}, t) \approx -\frac{q z_0 \omega^2}{4\pi\epsilon_0 c^3 r}\cos[\omega(t - r/c)]\sin\theta\,\hat{\boldsymbol{\varphi}}. \tag{70}$$

Remark: The point charge has an electric dipole moment $\boldsymbol{p} = q\boldsymbol{r}_q$, and it can be seen that (69)-(70) coincide with the leading contribution to the fields of an arbitrary localized source which is the electric dipole contribution given by[5] $\boldsymbol{E}_{\rm rad}(\boldsymbol{r}, t) = (\mu_0/4\pi r)\hat{\boldsymbol{r}} \times [\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}(t - r/c)]$, $\boldsymbol{B}_{\rm rad}(\boldsymbol{r}, t) = -(\mu_0/4\pi r)\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}(t - r/c)$ (when this contribution is nonzero). This therefore confirms the correctness of (69)-(70). More specifically, (69)-(70) coincide with the radiation fields of an electric dipole with a dipole moment in the z direction.[6] Other quantities derived from these (like $\boldsymbol{S}_{\rm rad}$; see below) will then also coincide.

(c) 1. The associated Poynting vector is

$$\boldsymbol{S}_{\rm rad}(\boldsymbol{r}, t) = \frac{1}{\mu_0}\boldsymbol{E}_{\rm rad}(\boldsymbol{r}, t) \times \boldsymbol{B}_{\rm rad}(\boldsymbol{r}, t) = \frac{q^2 z_0^2 \omega^4}{16\pi^2\epsilon_0 c^3}\cos^2[\omega(t - r/c)]\frac{\sin^2\theta}{r^2}\,\hat{\boldsymbol{r}},$$

where we used $1/\mu_0 = \epsilon_0 c^2$ and $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{r}}$. Using $\langle\cos^2[\omega(t - r/c)]\rangle = 1/2$ the time average is

$$\langle \boldsymbol{S}_{\rm rad}(\boldsymbol{r})\rangle = \frac{q^2 z_0^2 \omega^4}{32\pi^2\epsilon_0 c^3}\frac{\sin^2\theta}{r^2}\,\hat{\boldsymbol{r}}. \tag{70}$$

2. The power radiated through a sphere of radius r is $P(t) = \int \boldsymbol{S}_{\rm rad}(\boldsymbol{r}, t)\cdot d\boldsymbol{a}$. Thus the energy U radiated during a period $T = 2\pi/\omega$ is

$$U = \int_0^T dt\, P(t) = T\langle P\rangle = T\int \langle \boldsymbol{S}_{\rm rad}(\boldsymbol{r})\rangle \cdot d\boldsymbol{a} = \frac{2\pi}{\omega}\cdot\frac{q^2 z_0^2 \omega^4}{32\pi^2\epsilon_0 c^3}\underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\sin^2\theta}_{2\pi\cdot\frac{4}{3}} = \frac{q^2 z_0^2 \omega^3}{6\epsilon_0 c^3}. \tag{71}$$

[5]Discussed in 26/3 (replacement) lecture.
[6]Discussed in 25/3 lecture.

Appendix: Incorrect claims in answers to 2(c)1

Looking through the exam answers to question 2(c)1, three incorrect claims were common. These seem to be tied to mathematical issues, not physics. I hope the following discussion of them can be helpful.

The first incorrect claim is

$$\left\langle \frac{\partial \boldsymbol{S}}{\partial t} \right\rangle = \frac{\partial}{\partial t} \langle \boldsymbol{S} \rangle \qquad \text{(a logically incorrect statement)} \tag{72}$$

Both sides of this equation are 0, so in a trivial numerical sense the equation is true. But the rhs is not a logical consequence of the lhs. On the rhs, the time average $\langle \boldsymbol{S} \rangle$ is time-independent by construction, so taking its time derivative must necessarily give 0. The rhs seems to have been "obtained" from the lhs side by pulling the time derivative outside of the integral. But this can not be done, since the integration is also over the time variable, so the time derivative and integration do not commute. (In contrast, if the integral had instead been over spatial variables, one could have pulled the time derivative outside.) Another way to see this is that such commutation, if valid, should then also have been valid if the integration limits were changed to something else than an integral over a period. In that case the rhs would continue to be 0 (since the integral still produces a time-independent quantity), but the lhs would then in general not be 0, giving also a numerical contradiction.

The second incorrect claim is that $\boldsymbol{S}$ itself is time-independent. That this is incorrect can be seen explicitly from (40). This conclusion seems to have been reached by erroneously thinking that $\boldsymbol{S}$ itself, and not merely its time-average, can be found by using (18).

The third incorrect claim can be phrased as follows: Given that $\tilde{C}$ and $\tilde{D}$ are the complex quantities representing the real quantities C and D, then $\tilde{C}\tilde{D}$ is the complex quantity representing CD. (In the present context this claim took the form that $\boldsymbol{S}_T$ has the complex representation $(1/\mu_0)\tilde{\boldsymbol{E}}_T \times \tilde{\boldsymbol{B}}_T$.) That this claim is incorrect can be seen from a simple example. Suppose

$$\begin{aligned} C &= C_0 \cos(kz - \omega t + \delta_C) \quad \Rightarrow \quad \tilde{C} = \tilde{C}_0 e^{i(kz-\omega t)} \quad (\text{where } \tilde{C}_0 = C_0 e^{i\delta_C}), & (73) \\ D &= D_0 \cos(kz - \omega t + \delta_D) \quad \Rightarrow \quad \tilde{D} = \tilde{D}_0 e^{i(kz-\omega t)} \quad (\text{where } \tilde{D}_0 = D_0 e^{i\delta_C}). & (74) \end{aligned}$$

Thus

$$\tilde{C}\tilde{D} = C_0 D_0 e^{i(2kz - 2\omega t + \delta_C + \delta_D)} \tag{75}$$

so

$$\mathrm{Re}(\tilde{C}\tilde{D}) = C_0 D_0 \cos(2kz - 2\omega t + \delta_C + \delta_D), \tag{76}$$

which is not equal to the physical, real, quantity

$$CD = C_0 D_0 \cos(kz - \omega t + \delta_C) \cos(kz - \omega t + \delta_D). \tag{77}$$

The discrepancy occurs because the real part of the product of two complex quantities does not only get a contribution from the product of the individual real parts (which is the contribution we want) but also gets a contribution from the product of the two individual imaginary parts. You can see both contributions by using the formula for the cosine of a sum to rewrite the cosine on the rhs of (76) as $\cos(kz - \omega t + \delta_C)\cos(kz - \omega t + \delta_D) - \sin(kz - \omega t + \delta_C)\sin(kz - \omega t + \delta_D)$.