TFY4240 Electromagnetic theory: Solution to exam, spring 2022

Equation numbers in the exam text will be referred to with the prefix ET. Thus, for example, (ET4) refers to Eq. (4) in the exam text.

Problem 1

(a) 1. The infinitesimal charge dq_1 on an infinitesimal area da around the point $\boldsymbol{r}$ on the surface a of the segment of cylinder 1 is $dq_1 = \sigma_1(\boldsymbol{r})da$. As the force on that infinitesimal charge due to cylinder 2 is $d\boldsymbol{F} = dq_1\boldsymbol{E}_2(\boldsymbol{r})$, we get

$$\boldsymbol{F} = \int d\boldsymbol{F} = \int dq_1 \boldsymbol{E}_2(\boldsymbol{r}) = \int_a da\, \sigma_1(\boldsymbol{r})\boldsymbol{E}_2(\boldsymbol{r}). \tag{1}$$

In other words, to find $\boldsymbol{F}$ we can integrate $\sigma_1(\boldsymbol{r})\boldsymbol{E}_2(\boldsymbol{r})$ over the surface of the segment of cylinder 1.

2. Clearly the charge distribution on each cylinder has cylindrical symmetry around the cylinder axis. Due to this symmetry and the properties of the electric field in electrostatics (divergence proportional to ρ, and zero curl), it follows that[1] the electric field of cylinder α must point radially, with its magnitude only depending on the distance s_α to the cylinder. Thus we can write $\boldsymbol{E}_\alpha = E_\alpha \hat{\boldsymbol{s}}_\alpha$ where E_α (which is not necessarily positive the way we have defined it) only depends on s_α. Due to the high symmetry it is convenient to use Gauss's law in integral form, i.e.

$$\oint \boldsymbol{E}_\alpha \cdot d\boldsymbol{a} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \tag{2}$$

with the closed Gaussian surface a chosen to be a cylinder of radius s_α and length L that is concentric with the physical cylinder. First consider the surface integral on the LHS. There is no contribution from the two flat parts of the Gaussian cylinder surface since $d\boldsymbol{a}$ there is perpendicular to $\boldsymbol{E}_\alpha$. This leaves the contribution from the curved part, which, using $d\boldsymbol{a} = da\hat{\boldsymbol{s}}_\alpha$, is given by $E_\alpha \cdot 2\pi s_\alpha L$. On the other hand, Q_{enclosed} equals $\sigma_\alpha \cdot 2\pi bL$ for $s_\alpha > b$ and 0 for $s_\alpha < b$. Thus the field vanishes inside the cylinder ($s_\alpha < b$), while outside the cylinder ($s_\alpha > b$),

$$E_\alpha = \frac{b\sigma_\alpha}{\epsilon_0 s_\alpha}. \tag{3}$$

It follows from this result that outside the cylinder the field magnitude is given by Eq. (ET1), and that the field points in the outward direction (i.e. away from the cylinder axis) if $\sigma_\alpha > 0$ and in the inward direction (i.e. towards the cylinder axis) if $\sigma_\alpha < 0$.

(b) The second term in (ET2) vanishes. This can be argued either from the fact that this is a statics problem, so the time derivative must give 0, or from the fact that $\boldsymbol{B} = 0$ here, so $\boldsymbol{S} = 0$. Thus $\boldsymbol{F}$ simplifies to

$$\boldsymbol{F} = \oint_a \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} \tag{4}$$

and the components of the Maxwell stress tensor $\overleftrightarrow{\boldsymbol{T}}$ simplify to

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}E^2 \right). \tag{5}$$

The volume Ω can be chosen as described in the exam text because it satisfies the criteria: (i) it encloses all the charges we want to find the force on, (ii) it does not enclose any other charges. The volume Ω takes the form of a "half-cylinder" of length L and radius R. Thus its surface a consists of four parts: two half-disks of radius R, respectively at $z = L/2$ and $z = -L/2$, and two "strips" of width L, one of them curved (at distance R from the origin) and the other one flat (at $x = 0$, of height $2R$). We now consider their contributions to the surface integral in (4):

[1] More details about the reasoning leading to these conclusions could be given, but I omit them here.

- For the two half-disks at $z = \pm L/2$, note that for each point $(x, y, L/2)$ on the half-disk with $z = L/2$ and $d\boldsymbol{a} = da\hat{\boldsymbol{z}}$, there is an "opposite" point $(x, y, -L/2)$ on the half-disk with $z = -L/2$ and $d\boldsymbol{a} = da(-\hat{\boldsymbol{z}})$. As $\overleftrightarrow{\boldsymbol{T}}$ is identical at these points (because $\boldsymbol{E}$ is independent of z), the contributions $\overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a}$ from such pairs of points cancel. Thus the two half-disks give zero net contribution.

- For the curved strip, we can for $R \gg w$ approximate s_α in (ET1) by R, from which it follows that $\boldsymbol{E}$ decays like[2] $1/R$. Thus, since $\overleftrightarrow{\boldsymbol{T}}$ is quadratic in $\boldsymbol{E}$, $\overleftrightarrow{\boldsymbol{T}}$ decays like $1/R^2$. On the other hand, the area of the curved strip is $2\pi RL$ and thus grows like R. Therefore the contribution to $\oint_a \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a}$ scales like $1/R^2 \cdot R = 1/R$, which goes to 0 as $R \to \infty$. Thus the curved strip does not contribute to $\boldsymbol{F}$ in the limit $R \to \infty$.

Thus the only contribution comes from the flat strip at $x = 0$. We have (also using the Einstein summation convention)

$$\overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} = (T_{ij}\hat{\boldsymbol{x}}_i \otimes \hat{\boldsymbol{x}}_j) \cdot (da_k \hat{\boldsymbol{x}}_k) = T_{ij}\, da_k\, \hat{\boldsymbol{x}}_i\, (\hat{\boldsymbol{x}}_j \cdot \hat{\boldsymbol{x}}_k) = T_{ij}\, da_k\, \hat{\boldsymbol{x}}_i\, \delta_{jk} = T_{ik} da_k \hat{\boldsymbol{x}}_i. \tag{6}$$

Thus

$$F_x = \oint_a (\overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a})_x = \oint_a T_{xk} da_k = \oint_a (T_{xx} da_x + T_{xy} da_y + T_{xz} da_z). \tag{7}$$

For the flat strip at $x = 0$ we have $d\boldsymbol{a} = da \cdot (+\hat{\boldsymbol{x}})$, so $da_x = da$ and $da_y = da_z = 0$, where $da = dy\, dz$. Thus only T_{xx} contributes. Because nothing depends on z, the z-integration just gives the factor L. Furthermore, since $E_z = 0$, we get $T_{xx} = \epsilon_0(E_x^2 - (1/2)E^2) = (\epsilon_0/2)(E_x^2 - E_y^2)$. Thus

$$F_x = \frac{L\epsilon_0}{2} \int_{-\infty}^{\infty} dy\ (E_x^2 - E_y^2). \tag{8}$$

Here, $E_x = E_{1,x} + E_{2,x}$ and $E_y = E_{1,y} + E_{2,y}$ are at $x = 0$ given by

$$E_x = \frac{b}{\epsilon_0 s}(\sigma_1 - \sigma_2)\cos\theta, \tag{9}$$

$$E_y = \frac{b}{\epsilon_0 s}(\sigma_1 + \sigma_2)\sin\theta, \tag{10}$$

where $\cos\theta = w/s$, $\sin\theta = y/s$, with $s = \sqrt{w^2 + y^2}$. Inserting into (8) gives

$$F_x = \frac{Lb^2}{2\epsilon_0}\left[w^2(\sigma_1 - \sigma_2)^2 \int_{-\infty}^{\infty} dy\, \frac{1}{(w^2 + y^2)^2} - (\sigma_1 + \sigma_2)^2 \int_{-\infty}^{\infty} dy\, \frac{y^2}{(w^2 + y^2)^2}\right]. \tag{11}$$

Looking up the integrals, the first is $\pi/(2w^3)$, the second $\pi/(2w)$. After cancelling some terms, one arrives at

$$F_x = -\frac{\pi b^2 L \sigma_1 \sigma_2}{\epsilon_0 w}. \tag{12}$$

Remark: It can be checked that the same result is obtained by evaluating the integral (1), as it of course should be.

Problem 2

(a) 1. To show that $\tilde{\boldsymbol{E}}$ is perpendicular to $\boldsymbol{k}$, we can e.g. start from Gauss's law for $\boldsymbol{D}$, i.e. $\nabla \cdot \boldsymbol{D} = \rho_f$. Since there is no free charge, $\rho_f = 0$. Also, $\boldsymbol{D} = \epsilon \boldsymbol{E}$ in the simple medium (where the value of ϵ is for the frequency ω under consideration). This gives $\nabla \cdot (\epsilon \boldsymbol{E}) = 0$, which implies $\nabla \cdot \boldsymbol{E} = 0$. This also holds for the complex version of the electric field, i.e. we have

$$\nabla \cdot \tilde{\boldsymbol{E}} = 0. \tag{13}$$

[2] A "fine point": This $1/R$ decay is for $\sigma_1 + \sigma_2 \neq 0$. For the special case $\sigma_1 + \sigma_2 = 0$, the $1/R$ decays from the two cylinders will cancel, so the actual decay will be even faster (this is reminiscent of what happens if the monopole term for a *localized* charge distribution vanishes; note that our charge distribution extends to infinity and is thus *not* localized).

Evaluating the LHS of (13) for the plane wave gives (using Einstein's summation convention)

$$\partial_j \tilde{E}_j = \tilde{E}_{0,j} \partial_j e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = \tilde{E}_{0,j}\, ik_j e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = i\boldsymbol{k}\cdot\tilde{\boldsymbol{E}}. \tag{14}$$

Setting this equal to the RHS of (13) then gives

$$\boldsymbol{k}\cdot\tilde{\boldsymbol{E}} = 0, \tag{15}$$

i.e. $\tilde{\boldsymbol{E}}$ is perpendicular to $\boldsymbol{k}$.

To show that $\boldsymbol{B}$ is perpendicular to $\boldsymbol{k}$, we can e.g. start from Gauss's law for $\boldsymbol{B}$, i.e. $\nabla\cdot\boldsymbol{B} = 0$, which also holds for the complex version of the magnetic field, i.e.

$$\nabla\cdot\tilde{\boldsymbol{B}} = 0. \tag{16}$$

The evaluation of the LHS of (16) is done in exactly the same way as for the LHS of (13). Thus it immediately follows that also $\tilde{\boldsymbol{B}}$ is perpendicular to $\boldsymbol{k}$.

To show (ET4), we start from Faraday's law as applied to the complex versions of the fields:

$$\nabla\times\tilde{\boldsymbol{E}} = -\frac{\partial\tilde{\boldsymbol{B}}}{\partial t}. \tag{17}$$

Now consider the ℓ'th component of this equation. Its LHS is (here $\epsilon_{\ell nm}$ is the Levi-Civita symbol)

$$\epsilon_{\ell mn}\partial_m \tilde{E}_n = \epsilon_{\ell mn}\tilde{E}_{0,n}\partial_m e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = \epsilon_{\ell mn}\tilde{E}_{0,n} ik_m e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = i\epsilon_{\ell mn}k_m\tilde{E}_n = i(\boldsymbol{k}\times\tilde{\boldsymbol{E}})_\ell, \tag{18}$$

while its RHS is $i\omega\tilde{\boldsymbol{B}}_\ell$. Equating these gives (ET4).

Remarks:

- Note that (ET4) implies the property $\tilde{\boldsymbol{B}} \perp \boldsymbol{k}$. This is therefore an alternative proof of this property.
- Similarly, an alternative proof of the property $\tilde{\boldsymbol{E}} \perp \boldsymbol{k}$ could be given as follows: Start from the Maxwell equation $\nabla\times\boldsymbol{H} = \boldsymbol{j}_f + \partial\boldsymbol{D}/\partial t$, with $\boldsymbol{j}_f = 0$, $\boldsymbol{D} = \epsilon\boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}/\mu$. Applying this to the complex versions of the fields, and working out the derivatives in the same way as for Faraday's law above, one finds that $\tilde{\boldsymbol{E}}$ is proportional to $\boldsymbol{k}\times\tilde{\boldsymbol{B}}$, from which the desired property follows.
- In the calculations above, I worked out the divergences and curls in terms of the vector components involved. Alternatively, one can work out these quantities using the vector identities (5) and (7), respectively, in the general formula set.

2. (Being very elementary and not requiring any course-specific knowledge, this question was given a very low weight. It was included to help put you on the right track for the next question.)

$$\tilde{\boldsymbol{B}} \;=\; \frac{1}{\omega}(\boldsymbol{k}\times\tilde{\boldsymbol{E}}) = \frac{k}{\omega}\tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}(\cos\theta\,\hat{\boldsymbol{z}} + \sin\theta\,\hat{\boldsymbol{x}})\times\hat{\boldsymbol{y}} = \frac{1}{v}\tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}(-\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{z}}). \tag{19}$$

(b) Some general remarks: (i) $\boldsymbol{E}_1 = \boldsymbol{E}_I + \boldsymbol{E}_R$, $\boldsymbol{E}_2 = \boldsymbol{E}_T$, and ditto for $\boldsymbol{B}$. (ii) In this system, the component perpendicular ($\perp$) to the interface is the z component, and the part parallel ($\|$) to the interface is spanned by the x and y components.

Each boundary condition (BC) can be applied to the complex version of the fields. We will first consider the two boundary conditions (BC's) involving the electric field. We note that the electric field of all three waves (I, R, T) points along the y axis.

The BC $\epsilon\tilde{E}_1^\perp = \epsilon_2\tilde{E}_2^\perp$ becomes $\epsilon_1\tilde{E}_1^z = \epsilon_2\tilde{E}_2^z$. Thus since $\tilde{E}^z = 0$ for all three waves, this BC reduces to $0 = 0$ and is therefore trivially satisfied, giving no information about how the three waves are related.

The BC $\tilde{\boldsymbol{E}}_1^{\|} = \tilde{\boldsymbol{E}}_2^{\|}$ simplifies to $\tilde{E}_1^y = \tilde{E}_2^y$. Using the given fact that the exp(...) factors are identical for all three waves at the interface, this further simplifies to

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}. \tag{20}$$

Next we consider the magnetic field. First, from the figure in the exam text one can see that

$$\begin{aligned} \boldsymbol{k}_I &= k_I(\cos\theta_I\ \hat{\boldsymbol{z}} + \sin\theta_I\ \hat{\boldsymbol{x}}), & (21)\\ \boldsymbol{k}_R &= k_R(-\cos\theta_R\ \hat{\boldsymbol{z}} + \sin\theta_R\ \hat{\boldsymbol{x}}), & (22)\\ \boldsymbol{k}_T &= k_T(\cos\theta_T\ \hat{\boldsymbol{z}} + \sin\theta_T\ \hat{\boldsymbol{x}}), & (23)\end{aligned}$$

where $k_w = |\boldsymbol{k}_w| > 0$. Using (ET4) and adapting (19) (beware the sign change resulting from the *negative* z component of $\boldsymbol{k}_R$) gives

$$\begin{aligned} \tilde{\boldsymbol{B}}_I &= = \frac{1}{v_1}\tilde{E}_{0I}e^{i(\boldsymbol{k}_I\cdot\boldsymbol{r}-\omega_I t)}(-\cos\theta_I\ \hat{\boldsymbol{x}} + \sin\theta_I\ \hat{\boldsymbol{z}}), & (24)\\ \tilde{\boldsymbol{B}}_R &= = \frac{1}{v_1}\tilde{E}_{0R}e^{i(\boldsymbol{k}_R\cdot\boldsymbol{r}-\omega_R t)}(\cos\theta_R\ \hat{\boldsymbol{x}} + \sin\theta_R\ \hat{\boldsymbol{z}}), & (25)\\ \tilde{\boldsymbol{B}}_T &= = \frac{1}{v_2}\tilde{E}_{0T}e^{i(\boldsymbol{k}_T\cdot\boldsymbol{r}-\omega_T t)}(-\cos\theta_T\ \hat{\boldsymbol{x}} + \sin\theta_T\ \hat{\boldsymbol{z}}). & (26)\end{aligned}$$

Now we can use these expressions to consider the two BC's for the magnetic field. The BC $\tilde{B}_1^{\perp} = \tilde{B}_2^{\perp}$ becomes $\tilde{B}_1^z = \tilde{B}_2^z$. (Although this BC will not result in additional information (cf. the remark in the exam text), I consider it here for completeness.) Using that the exp-factors are identical for all three waves at the interface, this BC simplifies to

$$\frac{1}{v_1}(\tilde{E}_{0I}\sin\theta_I + \tilde{E}_{0R}\sin\theta_R) = \frac{1}{v_2}\tilde{E}_{0T}\sin\theta_T. \tag{27}$$

Using $\theta_R = \theta_I$, $n_1\sin\theta_I = n_2\sin\theta_T$ and $v_1/v_2 = n_2/n_1$, this expression simplifies to (20).

Finally, the BC $\tilde{\boldsymbol{B}}_1^{\|} = \tilde{\boldsymbol{B}}_2^{\|}$ simplifies to $\tilde{B}_1^x = \tilde{B}_2^x$. Again using the identity of the exp-factors, this further simplifies to

$$\frac{1}{v_1}(-\tilde{E}_{0I}\cos\theta_I + \tilde{E}_{0R}\cos\theta_R) = -\frac{1}{v_2}\tilde{E}_{0T}\cos\theta_T. \tag{28}$$

Using $\theta_R = \theta_I$, $v_1/v_2 = n_2/n_1$, and the definitions of α and β, this gives

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \alpha\beta\tilde{E}_{0T}. \tag{29}$$

(c) Adding and subtracting (20) and (29) gives the two equations $2\tilde{E}_{0I} = (1+\alpha\beta)\tilde{E}_{0T}$ and $2\tilde{E}_{0R} = (1-\alpha\beta)\tilde{E}_{0T}$, and dividing the former equation by the latter gives

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{1-\alpha\beta}{1+\alpha\beta}. \tag{30}$$

We see that the reflected wave would vanish if the ratio on the RHS were to vanish. Furthermore, writing $\tilde{E}_{0I} = |\tilde{E}_{0I}|e^{i\delta_I}$ and $\tilde{E}_{0R} = |\tilde{E}_{0R}|e^{i\delta_R}$, the LHS becomes

$$\frac{|\tilde{E}_{0R}|}{|\tilde{E}_{0I}|}e^{i(\delta_R-\delta_I)}, \tag{31}$$

so the phase difference $\delta_R - \delta_I$ (defined up to an integer multiple of 2π) is also determined by the ratio on the RHS of (30).

We must therefore consider the product $\alpha\beta$ which determines this ratio. We are given that n_1 and n_2 are both real and ≥ 1, with $n_1 < n_2$. It follows that $\beta = n_2/n_1$ is real and > 1. Furthermore, $\sin\theta_T = (n_1/n_2)\sin\theta_I < (n_1/n_2)\cdot 1 < 1$ so θ_T can interpreted as a conventional physical angle for all θ_I. Moreover, θ_T will lie between 0 and $\theta_{T,\max} = \arcsin(n_1/n_2) < \pi/2$, so $\cos\theta_T$ is a positive number (equal to $\cos\theta_T = \sqrt{1-\sin^2\theta_T}$), and thus $\alpha = \cos\theta_T/\cos\theta_I$ is real and positive. Thus $\alpha\beta$ is a positive real number.

It follows that the ratio on the RHS of (30) is always real. Thus the phase difference $\delta_R - \delta_I$ can only take the values 0 (this happens if the ratio is positive) or π (this happens if the ratio is negative), corresponding to the reflected and incident waves being "in phase" and "out of phase", respectively. (If the ratio is 0, the reflected wave vanishes, so the phase difference is not defined then.)

Furthermore, since the denominator $1+\alpha\beta$ is always positive, the sign of the ratio is the same as the sign of the numerator $1-\alpha\beta$, and the ratio would vanish if and only if the numerator vanishes. Thus we arrive at the following scenario:

- if $\alpha\beta > 1$, the phase difference is π
- if $\alpha\beta = 1$, the reflected wave vanishes
- if $\alpha\beta < 1$, the phase difference is 0.

We have

$$\alpha\beta \quad = \quad \frac{\cos\theta_T}{\cos\theta_I}\frac{n_2}{n_1} = \frac{\sqrt{1-\sin^2\theta_T}}{\sqrt{1-\sin^2\theta_I}}\frac{n_2}{n_1} = \frac{\sqrt{1-(n_1^2/n_2^2)\sin^2\theta_I}}{\sqrt{1-\sin^2\theta_I}}\frac{n_2}{n_1} = \sqrt{\frac{\beta^2-\sin^2\theta_I}{1-\sin^2\theta_I}}. \tag{32}$$

Consider the ratio under the square root. Since $\beta > 1$, the numerator is greater than the denominator, so the ratio is greater than 1, so the square root is greater than 1. Hence $\alpha\beta > 1$ for all θ_I. Therefore the phase difference is π for all θ_I, and the reflected wave never vanishes.

Remark: The angle(s) θ_I for which the reflected wave vanishes are referred to as *Brewster's angles*. Thus there are no Brewster angles in the setup considered here, for which the electric field is perpendicular to the plane of incidence.

Problem 3

(a) 1. The charge on the upper sphere is $\tilde{q}(t)$, and the charge on the lower sphere is $-\tilde{q}(t)$. Also, there is no net charge anywhere else (in particular, the wire is electrically neutral). Thus

$$\tilde{\rho}(\boldsymbol{r},t) = \tilde{q}(t)[\delta^{(3)}(\boldsymbol{r}-d\hat{\boldsymbol{z}}/2) - \delta^{(3)}(\boldsymbol{r}+d\hat{\boldsymbol{z}}/2)] = \tilde{q}(t)\delta(x)\delta(y)[\delta(z-d/2)-\delta(z+d/2)], \tag{33}$$

where δ is Dirac's delta function and $\delta^{(3)}$ its 3-dimensional generalization.

2. From (33) it follows that

$$\frac{\partial\tilde{\rho}}{\partial t} = \frac{d\tilde{q}(t)}{dt}\delta(x)\delta(y)[\delta(z-d/2)-\delta(z+d/2)]. \tag{34}$$

From (ET18) it follows that

$$\begin{aligned}\nabla\cdot\tilde{\boldsymbol{j}} \quad &= \quad \frac{d\tilde{q}(t)}{dt}\nabla\cdot\hat{\boldsymbol{z}}\,\delta(x)\delta(y)\Theta(d/2-|z|) = \frac{d\tilde{q}(t)}{dt}\delta(x)\delta(y)\underbrace{\frac{\partial}{\partial z}\Theta(d/2-|z|)}_{=-\mathrm{sgn(z)}\delta(\mathrm{d/2-|z|})} \\ &= \quad -\frac{d\tilde{q}(t)}{dt}\delta(x)\delta(y)[\delta(z-d/2)-\delta(z+d/2)]. \end{aligned}\tag{35}$$

(Here a derivation of, or at least a motivation for, the expression for the z-derivative of $\Theta(d/2-|z|)$ should also be provided; this can be done in many different ways.) One sees that $\partial\tilde{\rho}/\partial t = -\nabla\cdot\tilde{\boldsymbol{j}}$, i.e.

$$\frac{\partial\tilde{\rho}}{\partial t} + \nabla\cdot\tilde{\boldsymbol{j}} = 0. \tag{36}$$

This is the so-called *continuity equation.* That it holds is not some kind of special property of the system under consideration; quite the contrary, it is true for *any* system, being an expression of (local) conservation of electric charge.

Remark: When writing down the expression for $\tilde{\rho}(t)$, we treated the metal spheres as points. This is also an essential assumption underlying the expression (ET18) for $\tilde{\boldsymbol{j}}$. That this assumption affects both $\tilde{\rho}$ and $\tilde{\boldsymbol{j}}$ is not surprising in light of the continuity equation relating these quantities.

(b) Using the standard Lorenz gauge expression Eq. (ET24) in the formula set, we have

$$\tilde{V}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\tilde{\rho}(\boldsymbol{r'},t_{\text{ret}})}{|\boldsymbol{r}-\boldsymbol{r'}|}. \tag{37}$$

where $t_{\text{ret}} = t - |\boldsymbol{r}-\boldsymbol{r'}|/c$ is the retarded time. Inserting (33) gives

$$\tilde{V}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\tilde{q}(t_{\text{ret}})\delta(x')\delta(y')[\delta(z'-d/2)-\delta(z'+d/2)]}{|\boldsymbol{r}-\boldsymbol{r'}|}. \tag{38}$$

Carrying out the integral (which is simple due to the Dirac delta functions) gives

$$\tilde{V}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{\tilde{q}(t-|\boldsymbol{r}-d\hat{\boldsymbol{z}}/2|/c)}{|\boldsymbol{r}-d\hat{\boldsymbol{z}}/2|} - \frac{\tilde{q}(t-|\boldsymbol{r}+d\hat{\boldsymbol{z}}/2|/c)}{|\boldsymbol{r}+d\hat{\boldsymbol{z}}/2|}\right]. \tag{39}$$

Here

$$\tilde{q}(t-|\boldsymbol{r}\mp d\hat{\boldsymbol{z}}/2|/c) = q_0\exp[-i\omega(t-|\boldsymbol{r}\mp d\hat{\boldsymbol{z}}/2|/c)] = q_0 e^{-i\omega t}e^{ik|\boldsymbol{r}\mp d\hat{\boldsymbol{z}}/2|}. \tag{40}$$

Thus $|\boldsymbol{r}\mp d\hat{\boldsymbol{z}}/2|$ appears both in the exponent and in the denominator. We have

$$|\boldsymbol{r}\mp d\hat{\boldsymbol{z}}/2| = \sqrt{(\boldsymbol{r}\mp d\hat{\boldsymbol{z}}/2)\cdot(\boldsymbol{r}\mp d\hat{\boldsymbol{z}}/2)} = \sqrt{r^2 \mp \boldsymbol{r}\cdot\hat{\boldsymbol{z}}d + d^2/4} = r\sqrt{1\mp \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}}\frac{d}{r} + \left(\frac{d}{2r}\right)^2}. \tag{41}$$

Since $d \ll r$, we try to expand the square root in the small parameter d/r. The most naive approximation one might try is to only use the zeroth order term, which corresponds to replacing $|\boldsymbol{r}\mp d\hat{\boldsymbol{z}}/2|$ by r. However, making this replacement both in the exponent and in the denominator causes the two terms in (39) to cancel each other, leaving $\tilde{V}=0$, while we would expect that $\tilde{V}$ would have to decay like $1/r$ in order to contribute to the radiation fields. Thus we must try a less naive approximation. We try a first-order expansion in d/r as the simplest next approximation. Using $(1+u)^n \approx 1+nu$ for $u \ll 1$ gives

$$|\boldsymbol{r}\mp d\hat{\boldsymbol{z}}/2| \approx r\left(1\mp\frac{1}{2}\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}}\frac{d}{r}\right) = r\mp\frac{d}{2}\cos\theta, \tag{42}$$

where we used $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}} = \cos\theta$. Using this first-order expansion in the exponent, while keeping the simplest, zeroth-order expansion in the denominator, gives

$$\tilde{V}(\boldsymbol{r},t) \approx \frac{1}{4\pi\epsilon_0}q_0 e^{-i\omega t}\cdot\frac{1}{r}\cdot e^{ikr}\left[e^{-ik(d/2)\cos\theta} - e^{ik(d/2)\cos\theta}\right] = -\frac{iq_0}{2\pi\epsilon_0}\sin\left(\frac{kd}{2}\cos\theta\right)\frac{e^{i(kr-\omega t)}}{r}. \tag{43}$$

(Including the first-order term also in the expansion of the denominator would give corrections to $\tilde{V}$ that decay like $1/r^2$, which would not contribute to radiation.)

Remark: It can be checked that the same approximation scheme as used here can also be used to derive Eq. (ET19) for $\tilde{A}(\boldsymbol{r},t)$.

(c) We use $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. To calculate the curl, it is most convenient to make use of the general expression for the curl in a particular coordinate system. We see that $\tilde{\boldsymbol{A}}$ involves the coordinates r and θ associated with the spherical coordinate system, as well as the unit vector $\hat{\boldsymbol{z}}$ associated with the cylindrical coordinate system. It will be seen that using spherical coordinates gives the simplest calculation, so I consider that before cylindrical coordinates. (The general expressions for the curl in spherical and cylindrical coordinates are given in the general formula set.)

Spherical coordinates. In the general formula set we see that $\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$. Inserting this into Eq. (ET19) gives an expression for $\tilde{\boldsymbol{A}}$ involving only spherical coordinates. We get

$$\tilde{\boldsymbol{A}} = \tilde{A}_r\hat{\boldsymbol{r}} + \tilde{A}_\theta\hat{\boldsymbol{\theta}} + \tilde{A}_\phi\hat{\boldsymbol{\phi}} \tag{44}$$

where

$$\begin{aligned}
\tilde{A}_r &= -\frac{icq_0\mu_0}{2\pi}\sin\left(\frac{kd}{2}\cos\theta\right)\frac{\exp(i(kr-\omega t))}{r}, && (45)\\
\tilde{A}_\theta &= \frac{icq_0\mu_0}{2\pi}\tan\theta\sin\left(\frac{kd}{2}\cos\theta\right)\frac{\exp(i(kr-\omega t))}{r}, && (46)\\
\tilde{A}_\phi &= 0. && (47)
\end{aligned}$$

Taking into account the form of these components (note that neither $\tilde{A}_r$ nor $\tilde{A}_\theta$ depend on ϕ), many terms in the general expression for the curl in spherical coordinates vanish, leaving

$$\nabla\times\tilde{\boldsymbol{A}} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\tilde{A}_\theta) - \frac{\partial\tilde{A}_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}. \tag{48}$$

One sees that the second term, which involves the factor $\partial\tilde{A}_r/\partial\theta$, decays like $1/r^2$ and will thus not contribute to radiation. This leaves the contribution from the first term, giving

$$\tilde{\boldsymbol{B}}_{\text{rad}}(\boldsymbol{r},t) = -\frac{ckq_0\mu_0}{2\pi}\tan\theta\sin\left(\frac{kd}{2}\cos\theta\right)\frac{\exp(i(kr-\omega t))}{r}\hat{\boldsymbol{\phi}}. \tag{49}$$

Cylindrical coordinates. Using that $\tilde{A}_s = \tilde{A}_\phi = 0$, the general expression for the curl in cylindrical coordinates simplifies to

$$\nabla\times\tilde{\boldsymbol{A}} = \frac{1}{s}\frac{\partial\tilde{A}_z}{\partial\phi}\hat{\boldsymbol{s}} - \frac{\partial\tilde{A}_z}{\partial s}\hat{\boldsymbol{\phi}}. \tag{50}$$

Beware that the partial derivatives here should be evaluated *with the other cylindrical coordinates held constant*, while our expression for $\tilde{A}_z$ instead depends on the spherical coordinates r and θ. One approach would be to first express $\tilde{A}_z$ in cylindrical coordinates by using the expressions for spherical coordinates in terms of cylindrical coordinates:

$$\begin{aligned}
r &= \sqrt{s^2+z^2}, && (51)\\
\cos\theta &= \frac{z}{\sqrt{s^2+z^2}}, && (52)\\
\phi &= \phi. && (53)
\end{aligned}$$

(Here I chose to use $\cos\theta$ instead of θ, as this will simplify some calculations.) However, this approach might give quite nasty expressions. Instead, we can express the partial derivatives as

$$\begin{aligned}
\frac{\partial\tilde{A}_z}{\partial\phi} &\equiv \frac{\partial\tilde{A}_z}{\partial\phi}\Big|_{s,z} = \frac{\partial r}{\partial\phi}\Big|_{s,z}\frac{\partial\tilde{A}_z}{\partial r}\Big|_{\theta,\phi} + \frac{\partial\cos\theta}{\partial\phi}\Big|_{s,z}\frac{\partial\tilde{A}_z}{\partial\cos\theta}\Big|_{r,\phi}, && (54)\\
\frac{\partial\tilde{A}_z}{\partial s} &\equiv \frac{\partial\tilde{A}_z}{\partial s}\Big|_{\phi,z} = \frac{\partial r}{\partial s}\Big|_{\phi,z}\frac{\partial\tilde{A}_z}{\partial r}\Big|_{\theta,\phi} + \frac{\partial\cos\theta}{\partial s}\Big|_{\phi,z}\frac{\partial\tilde{A}_z}{\partial\cos\theta}\Big|_{r,\phi}. && (55)
\end{aligned}$$

The expression in (54) vanishes because neither r nor θ depend on ϕ (this can be seen from (51) and (52); more simply it follows from the fact that ϕ is also a spherical coordinate and thus independent of r and θ). To work out the expression in (55) we use (51)-(52) to find

$$\begin{aligned}
\frac{\partial r}{\partial s}\Big|_{\phi,z} &= \frac{s}{\sqrt{s^2+z^2}} = \frac{r\sin\theta}{r} = \sin\theta, && (56)\\
\frac{\partial\cos\theta}{\partial s}\Big|_{\phi,z} &= z(s^2+z^2)^{-3/2}(-1/2)\cdot 2s = -\frac{sz}{(s^2+z^2)^{3/2}} = -\frac{r\sin\theta\, r\cos\theta}{r^3} = -\frac{\sin\theta\cos\theta}{r}. && (57)
\end{aligned}$$

Note that $\partial \cos\theta / \partial s|_{\phi,z}$ is proportional to $1/r$. As also $\partial \tilde{A}_z / \partial \cos\theta|_{r,\phi}$ can be seen to decay like $1/r$, their product will decay like $1/r^2$ and thus not contribute to radiation. Finally,

$$\left.\frac{\partial \tilde{A}_z}{\partial r}\right|_{\theta,\phi} = \frac{ckq_0\mu_0}{2\pi} \frac{\sin\left(\frac{kd}{2}\cos\theta\right)}{\cos\theta} \frac{\exp(i(kr-\omega t))}{r} \tag{58}$$

which was obtained by differentiating $\exp(i(kr-\omega t))$ (another contribution, $\propto 1/r^2$, follows by differentiating $1/r$ and is therefore neglected as it doesn't contribute to radiation). This leads to the expression (49) for $\tilde{\boldsymbol{B}}_{\text{rad}}$.

We see that the radiation fields $\tilde{\boldsymbol{E}}_{\text{rad}}(\boldsymbol{r},t)$ and $\tilde{\boldsymbol{B}}_{\text{rad}}(\boldsymbol{r},t)$ are outgoing spherical waves of frequency ω, in phase, with perpendicular directions, and magnitudes related by

$$\frac{|\tilde{\boldsymbol{B}}_{\text{rad}}|}{|\tilde{\boldsymbol{E}}_{\text{rad}}|} = \frac{c\mu_0}{1/\epsilon_0} = c\mu_0\epsilon_0 = \frac{1}{c} \tag{59}$$

(where we used $\epsilon_0\mu_0 = 1/c^2$). These properties are all as expected.

(d) As each field contributes the same angular factor $\tan\theta \sin((kd/2)\cos\theta)$, the angular dependence of the radiation pattern of the Poynting vector $\boldsymbol{S}$ (proportional to $\boldsymbol{E} \times \boldsymbol{B}$) becomes

$$\tan^2\theta \, \sin^2\left(\frac{kd}{2}\cos\theta\right). \tag{60}$$

We see that the radiation pattern is independent of ϕ, which is easy to understand from the symmetry of the system.

For $kd \ll 1$, we can use $\sin v \approx v$ for $v \ll 1$ to approximate $\sin((kd/2)\cos\theta) \approx (kd/2)\cos\theta$, so the angular dependence of the radiation pattern simplifies to

$$\tan^2\theta\cos^2\theta = \sin^2\theta. \tag{61}$$

The radiation is thus maximal for $\theta = \pi/2$ and minimal (vanishes) for $\theta = 0$ and π. When plotted as a function of θ and ϕ the pattern takes a donut shape.

We are familiar with this donut pattern from our investigation in the lectures of the radiation from an electric dipole. There we considered the limit of a point dipole from the outset, while in this problem we only took this limit at the end ($kd \ll 1$ being equivalent to $d \ll c/\omega$). It is therefore no surprise that we recover the same radiation pattern.