NTNU **Institutt for fysikk**

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Exam in TFY4240 Electromagnetic Theory

August 9, 2024
09:00–13:00

Allowed help: Alternativ **C**

A permitted basic calculator and a mathematical formula book (Rottmann or equivalent).

This problem set consists of 14 pages.

This exam consists of 4 problems, each containing several subproblems. There are in total 10 subproblems. Each subproblem (1a-1b-...) will be given equal weight in the grading.

The problems are given in English only. Do not hesitate to ask if you have any language problems related to the exam set. For your answers, you are free to use either English or Norwegian.

Some formulas are given in the appendix on the pages following the last problem.

Good luck!

Problem 1.

a) We consider the system shown in Fig. 1. The metals are grounded, and the potential

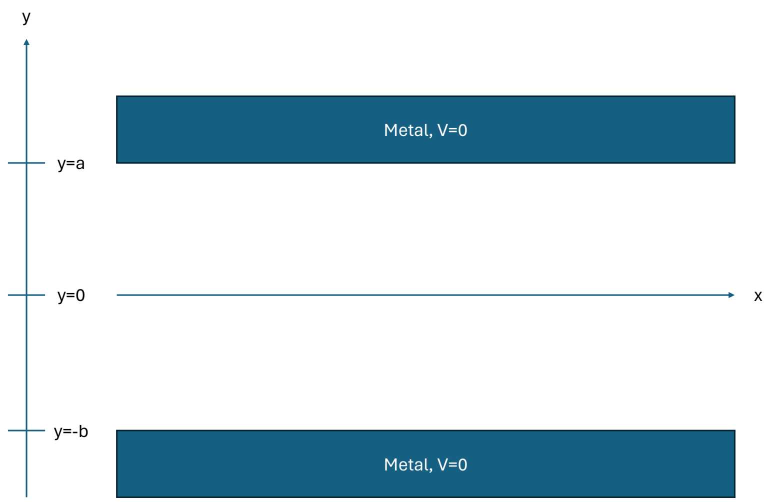

Figure 1: A sheet of surface charge at $y = 0$ is between two grounded metal conductors at $y = a$ and $y = -b$. Although not shown, the system is supposed to be infinitely homogenous in the x and z directions. At $y = 0$, there is a surface charge density $\sigma(x, t)$ that may depend on the spatial coordinate x and the time t..

is $V = 0$ therein. Between the metals, there is a homogenous surface charge density σ at $y = 0$. Above the surface charge density, when $0 < y < a$, the dielectric constant is ϵ_1. Below the surface charge density, when $0 > y > -b$, the dielectric constant is ϵ_2.

We assume the surface charge density is static and varies as $\sigma(x) = \sigma_0 \cos kx$.

Introduce the scalar potential V, solve the equation for V, and determine the electrostatic electric field $\boldsymbol{E} = -\boldsymbol{\nabla} V$ between the metals. Note that the electric field may be inhomogeneous.

Solution

Symmetry dictates that the potential V only depends on the coordinates y and x. From Gauss's law (99), we find the Laplace equation for the potential in between the metals when $y \neq 0$:

$$\left(\partial_x^2 + \partial_y^2\right) V(x, y) = 0 \,. \tag{1}$$

This is a separable differential equation. In general, the solution above (a) or below (b) the surface charges is a linear combination of contributions for all wave vectors κ, $V_i(x, y) = \int d\kappa V_{i\kappa}(x, y)$, where $i = a$ or $i = b$. Since we must have $V(x, y = a) = 0$

and $V(x, y = -b) = 0$ and since the electric fields must vary along the x direction as the surface charge density, we must have $\kappa = k$ and the solution must be of the form

$$V_a(x, y) = A \cos kx \sinh k\,(y - a)\,, \tag{2}$$
$$V_b(x, y) = B \cos kx \sinh k\,(y + b)\,. \tag{3}$$

The associated components of the electric field are then

$$E_{a,x}(x, y) = Ak \sin kx \sinh k\,(y - a)\,, \tag{4}$$
$$E_{b,x}(x, y) = Bk \sin kx \sinh k\,(y + b) \tag{5}$$

and

$$E_{a,y}(x, y) = -Ak \cos kx \cosh k\,(y - a) \tag{6}$$
$$E_{b,y}(x, y) = -Bk \cos kx \cosh k\,(y + b) \tag{7}$$

The boundary conditions are for the tangential field (105)

$$E_x(x, y = 0^+) - E_x(x, y = 0^-) = 0\,, \tag{8}$$
$$E_{a,x}(x, y = 0) - E_{b,x}(x, y = 0) = 0\,, \tag{9}$$
$$Ak \sin kx \sinh(-ka) - Bk \sin kx \sinh kb = 0 \tag{10}$$

which implies that

$$A = -B\frac{\sinh kb}{\sinh ka}\,. \tag{11}$$

The boundary conditions for the normal field component is (107)

$$\epsilon_1 E_y(x, y = 0^+) - \epsilon_2 E_y(x, y = 0^-) = \sigma_0 \cos kx \tag{12}$$
$$-\epsilon_1 Ak \cos kx \cosh(-ka) + \epsilon_2 Bk \cos kx \cosh kb = \sigma_0 \cos kx\,, \tag{13}$$
$$\left(\epsilon_1 \frac{\sinh kb}{\sinh ka} \cosh ka + \epsilon_2 \cosh kb\right) Bk = \sigma_0 \tag{14}$$

so that

$$A = -\frac{\sigma_0 \sinh kb}{k\,(\epsilon_1 \sinh kb \cosh ka + \epsilon_2 \cosh kb \sinh ka)}\,, \tag{15}$$
$$B = \frac{\sigma_0 \sinh ka}{k\,(\epsilon_1 \sinh kb \cosh ka + \epsilon_2 \cosh kb \sinh ka\,.)} \tag{16}$$

The computed coefficients A and B then determine the electric fields expressed in Eqs. (4), (5), (6), and (7).

b) We consider a material that is linear and isotropic.

The Poynting vector $\boldsymbol{S}$ and the electromagnetic energy density u are determined by

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{17}$$

$$u = \frac{1}{2}\left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{H}\right) . \tag{18}$$

Show that

$$\frac{\partial u}{\partial t} + \boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{\nabla} \cdot S = 0 . \tag{19}$$

Explain what this equation means and what the three terms on the left-hand side describe.

Solution

We use

$$\boldsymbol{\nabla} \cdot S = \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}) , \tag{20}$$

$$= \boldsymbol{H} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) \tag{21}$$

and insert Faraday‘s law (101) and Ampere‘s law (102) to find

$$\boldsymbol{\nabla} \cdot S = \boldsymbol{H} \cdot (-\partial_t \boldsymbol{B}) - \boldsymbol{E} \cdot (\boldsymbol{J} + \partial_t \boldsymbol{D}) , \tag{22}$$

$$= -\frac{1}{2}\partial_t \left(\frac{1}{\mu}\boldsymbol{B}^2 + \epsilon \boldsymbol{E}^2\right) - \boldsymbol{E} \cdot \boldsymbol{J} , \tag{23}$$

$$= -\partial_t u - \boldsymbol{E} \cdot \boldsymbol{J} \tag{24}$$

which is the equation we should prove.

The equation expresses the conservation of energy (electromagnetic and mechanical). For an infinitesimal volume element, the first term in Eq. (19) ($\partial_t u$) describes energy increase per unit time, the second term ($\boldsymbol{J} \cdot \boldsymbol{E}$) describes the power transformed to mechanical energy for the charges within the volume, and the last term ($\boldsymbol{\nabla} \cdot \boldsymbol{S}$) the energy flow out of the system.

Problem 2.

We consider vacuum. In the initial reference frame F1, the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ satisfy the wave equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\partial_t^2\right] \boldsymbol{E}(\boldsymbol{r}, t) = 0 , \tag{25}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\partial_t^2\right] \boldsymbol{B}(\boldsymbol{r}, t) = 0 . \tag{26}$$

The electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ are also related via Faraday's law (101). We consider another frame of reference F2 with spatial coordinate $\boldsymbol{R} = (X, Y, Z)$ and temporal coordinate τ that is related to the original reference frame with spatial coordinate $\boldsymbol{r} = (x, y, z)$ and temporal coordinate t by the Lorentz transformation

$$\tau = \gamma \left(t - vx/c^2\right) , \tag{27}$$
$$X = \gamma (x - vt) , \tag{28}$$
$$Y = y , \tag{29}$$
$$Z = z , \tag{30}$$

where

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}} \tag{31}$$

is the Lorentz factor. In other words, the initial reference frame F1 is moving with a velocity v along the x direction with respect to the other reference frame F2. We also have the inverse relationship

$$t = \gamma \left(\tau + vX/c^2\right) , \tag{32}$$
$$x = \gamma (X + v\tau) , \tag{33}$$
$$y = Y , \tag{34}$$
$$z = Z , \tag{35}$$

a) What are the wave equations for the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ in terms of the spatial coordinate $\boldsymbol{R}$ and temporal coordinate τ?

Solution

We transform the coordinates. For the spatial coordinates, we find

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} , \tag{36}$$
$$= \gamma\frac{\partial}{\partial X} - \gamma\frac{v}{c^2}\frac{\partial}{\partial \tau} , \tag{37}$$
$$\frac{\partial}{\partial Y} = \frac{\partial}{\partial y} , \tag{38}$$
$$\frac{\partial}{\partial Z} = \frac{\partial}{\partial z} . \tag{39}$$

For the temporal coordinate, we have

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial X}{\partial t}\frac{\partial}{\partial X} , \tag{40}$$
$$= \gamma\frac{\partial}{\partial \tau} - \gamma v\frac{\partial}{\partial X} . \tag{41}$$

We then find that

$$\frac{\partial^2}{\partial x^2} = \gamma^2 \frac{\partial^2}{\partial X^2} + \gamma^2 \frac{v^2}{c^4} \frac{\partial^2}{\partial \tau^2} - 2\gamma^2 \frac{v}{c^2} \frac{\partial}{\partial X} \frac{\partial}{\partial \tau}\,, \tag{42}$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial Y^2}\,, \tag{43}$$

$$\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial Z^2} \tag{44}$$

and

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \gamma^2 \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} + \gamma^2 \frac{v^2}{c^2} \frac{\partial^2}{\partial X^2} - 2\gamma^2 \frac{v}{c^2} \frac{\partial}{\partial X} \frac{\partial}{\partial \tau} \tag{45}$$

We then see that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\partial_t^2 = \gamma^2 \left(1 - \frac{v^2}{c^2}\right) \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \gamma^2 \left(1 - \frac{v^2}{c^2}\right) \frac{1}{c^2}\partial_\tau^2\,, \tag{46}$$

$$= \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} - \frac{1}{c^2}\partial_\tau^2\,. \tag{47}$$

This means that the wave equations for the electric field and the magnetic induction are Lorentz invariant, as should be expected:

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} - \frac{1}{c^2}\partial_\tau^2\right] \boldsymbol{E}(\boldsymbol{R}, \tau) = 0\,, \tag{48}$$

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} - \frac{1}{c^2}\partial_\tau^2\right] \boldsymbol{B}(\boldsymbol{R}, \tau) = 0\,. \tag{49}$$

b) Using complex notation, we consider a plane wave

$$\boldsymbol{E}(x, y, z, t) = E_0 \hat{\mathbf{y}} e^{i(kx - \omega t)}\,. \tag{50}$$

Compute the frequency Ω a person in reference F2 will observe in terms of the frequency ω in reference F1, the velocity v, and the velocity of light c.

Solution

We must transform the expression for the plane wave in the initial coordinates in reference frame F1 to an expression for the plane wave in reference frame F2. To this end, we use

$$kx - \omega t = k\gamma\,(X + vt) - \omega\gamma\left(\tau + vX/c^2\right)\,, \tag{51}$$

$$= \gamma\left(k - \omega\frac{v}{c^2}\right) X - \gamma\omega\left(1 - \frac{v}{c}\right)\,, \tag{52}$$

$$= Kx - \Omega t\,, \tag{53}$$

where we have used that $\omega = c$ and then find that

$$\Omega = \omega\sqrt{\frac{c-v}{c+v}} \tag{54}$$

and $\Omega = Kc$, which corresponds to the relativistic Doppler shift of the frequency. In reference frame F2, the plane wave is

$$\boldsymbol{E}(X, Y, Z, \tau) = E_0\hat{\mathbf{y}}e^{i(Kx-\Omega\tau)}\,. \tag{55}$$

Problem 3.

We consider two materials, 1 and 2, and the interface between them.

a) Derive the boundary conditions (105) and (106).

Solution

We start from Faraday‘s law and integrate over a small area S with a normal vector that is parallel to the interface:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \tag{56}$$

$$\int d\boldsymbol{S} \cdot \boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \int d\boldsymbol{S} \cdot \boldsymbol{B} \tag{57}$$

$$\oint d\boldsymbol{l}\boldsymbol{E} = -\partial_t \int d\boldsymbol{S} \cdot \boldsymbol{B}\,. \tag{58}$$

The surface area perpendicular to the interface is shown in Fig. 2

On the left-hand side of Eq. (58), we have converted the surface integral to a line integral around the surface using Stoke‘s theorem. Next, we let the height of the surface approach zero so that the area of the surface approaches zero on the right-hand side of (58), and the line integrals on the right-hand side only contain contributions from lines parallel to the interface. The result is valid for any surface area perpendicular to the interface and, hence, any parallel line. Hence, we find the boundary condition (105).

We can carry out a similar argument for the boundary condition for the magnetic field. We start from Ampere‘s law and integrate over a small area S with a normal vector that is parallel to the interface:

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f + \partial_t \boldsymbol{D}\,, \tag{59}$$

$$\int d\boldsymbol{S} \cdot \boldsymbol{\nabla} \times \boldsymbol{H} = \int d\boldsymbol{S} \cdot \boldsymbol{J}_f + \partial_t \int d\boldsymbol{S} \cdot \boldsymbol{D}\,, \tag{60}$$

$$\oint d\boldsymbol{l} \cdot \boldsymbol{H} = I_f + \partial_t \int d\boldsymbol{S} \cdot \boldsymbol{D}\,, \tag{61}$$

where I_f is the free charge current through the surface.

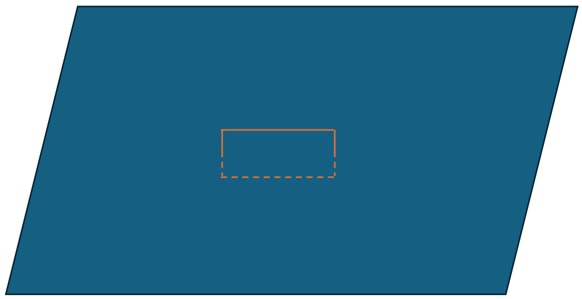

Figure 2: An interface between two materials. The surface denoted with straight (dashed) lines above (below) the interface is perpendicular to the interface.

Now, we do as above and let the height of the surface approach zero so that the area of the surface in the last term on the right-hand side of (61) vanishes. The result is valid for any surface area perpendicular to the interface and any parallel line. Hence, we find the boundary condition (106).

b) Derive the boundary conditions (107) and (108).

Solution

We start from Gauss‘ law and integrate over a small volume at the interface as shown in Fig. 3. We use the divergence theorem (114) to convert the integral on the left-hand side to a surface integral.

$$\boldsymbol{\nabla} \boldsymbol{D} = \rho \,, \tag{62}$$

$$\int d^3 r \boldsymbol{\nabla} \cdot \boldsymbol{D} = q \,, \tag{63}$$

$$\oint d\boldsymbol{S} \cdot \boldsymbol{D} = q \,, \tag{64}$$

where q is the total charge within the volume equal to the total charge at the surface when the surface height approaches zero. In the limit of zero height, we only have contributions from the normal vector of the displacement field on the left-hand side of Eq. (64). Hence, since this result is valid for any surface area, we find Eq. (107).

The proof of the boundary condition for the magnetic induction, Eq. (108) follows in a similar way from Gauss‘ law for magnetic induction, Eq. (100) with the only difference that there are no magnetic monopole charges.

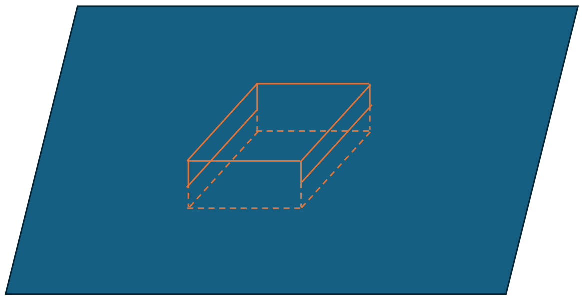

Figure 3: An interface between two materials. The volume is denoted by straight (dashed) lines above (below), and the interface has the largest areas parallel to it.

Problem 4.

a) We consider a microwave cavity as shown in Fig. 4. The cavity dimensions are a along the x direction, b along the y direction, and c along the z direction. Metallic plates enclose the cavity.

In the Lorentz gauge, and in the absence of free charges, the scalar potential V fulfills the wave equation

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right) V(x, y, z) = 0\,. \tag{65}$$

Choose the coordinate system so that one metal plate is located at $x = 0$ and another at $x = a$. Similarly, there are metal plates located at $y = 0$, $y = b$, $z = 0$, and $z = c$. What possible modes for the scalar potential V can exist inside the cavity?

Solution

The scalar potential must satisfy the wave equation (65) that in Cartesian coordinates becomes

$$\left(\partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{1}{c^2}\partial_t^2\right) V(x, y, z, t) = 0 \tag{66}$$

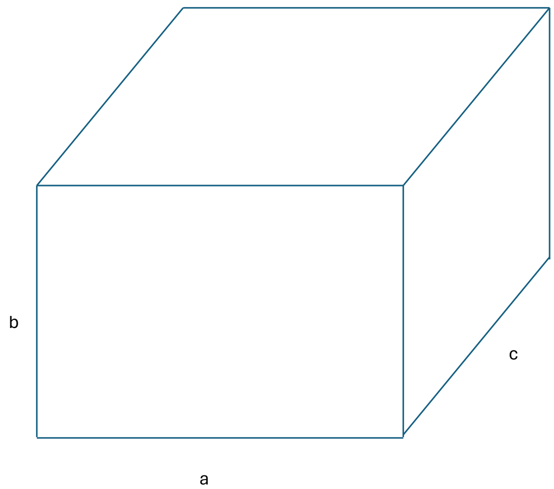

Figure 4: A microwave cavity. Metallic plates enclose the volume abc.

We use separation of variables, $V(x, y, z) = X(x)Y(y)Z(z)T(t)$ and find that

$$\frac{d^2}{dx}^2 X(x) = k_x^2 \,, \tag{67}$$

$$\frac{d^2}{dy}^2 Y(y) = k_y^2 \,, \tag{68}$$

$$\frac{d^2}{dz}^2 Z(z) = k_z^2 \,, \tag{69}$$

$$\frac{1}{c^2}\frac{d^2}{dt}^2 T(t) = k_x^2 + k_y^2 + k_z^2 \,, \tag{70}$$

where k_x, k_y, and k_z are constants. We define the frequency ω as

$$\omega = c\sqrt{k_x^2 + k_y^2 + k_z^2} \,. \tag{71}$$

A general solution is

$$X(x) = A\cos k_x x + B\sin k_x x \,, \tag{72}$$

$$Y(y) = C\cos k_y y + D\sin k_y y \,, \tag{73}$$

$$Z(z) = E\cos k_z z + F\sin k_z z \tag{74}$$

$$T(t) = G\cos\omega t + H\sin\omega t \tag{75}$$

for any values of the wave vectors k_x, k_y, and k_z.

The metal plates have the same potential, which we set to zero. That implies the boundary conditions

$$X(x=0)=0\,, \tag{76}$$
$$X(x=a)=0\,, \tag{77}$$
$$Y(y=0)=0\,, \tag{78}$$
$$Y(y=b)=0\,, \tag{79}$$
$$Z(z=0)=0\,, \tag{80}$$
$$Z(z=c)=0\,. \tag{81}$$

We then find that $A=0$, $B=0$, and $C=0$ and that the wave vectors are quantized so that

$$k_x=\frac{n_x\pi}{a} \tag{82}$$
$$k_y=\frac{n_y\pi}{b} \tag{83}$$
$$k_z=\frac{n_z\pi}{c}\,, \tag{84}$$

where $n_x=1,2,\ldots$, $n_y=1,2,\ldots$, and $n_z=1,2,\ldots$ are integral quantum numbers.

Without loss of generality, we may set $B=1$, $D=1$, and $F=1$ so that any potential is a linear combination of the eigenmode solutions

$$V_{n_X,n_y,n_z}(x,y,z,t)=\sin\frac{n\pi}{a}x\sin\frac{n\pi}{b}y\sin\frac{n\pi}{a}z\left(G_{n_x,n_y,n_z}\cos\omega_{n_x,n_y,n_z}t+H_{n_x,n_y,n_z}\sin\omega_{n_x,n_y,n_z}t\right) \tag{85}$$

where the eigenfrequency is

$$\omega_{n_x,n_y,n_z}=c\pi\sqrt{\left(\frac{n_x}{a}\right)^2+\left(\frac{n_y}{b}\right)^2+\left(\frac{n_z}{c}\right)^2}\,. \tag{86}$$

b) We consider a point charge q that is located at position $R=(0,a,0)$ above a grounded metal plate at $y=0$ as schematically shown in Fig. 5.

Compute the electric field as a function of position for all locations above the plane.

Solution

In the metal, the potential is constant, and we choose it to be zero. We use the method of image charges to enforce the condition that the potential should be constant in the plane defined by $y=0$. This implies that we introduce a fictitious charge $-q$ located at position $(0,-a,0)$. The potential is then

$$V(x,y,z)=\frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+(y-a)^2+z^2}}-\frac{1}{\sqrt{x^2+(y+a)^2+z^2}}\right)\,. \tag{87}$$

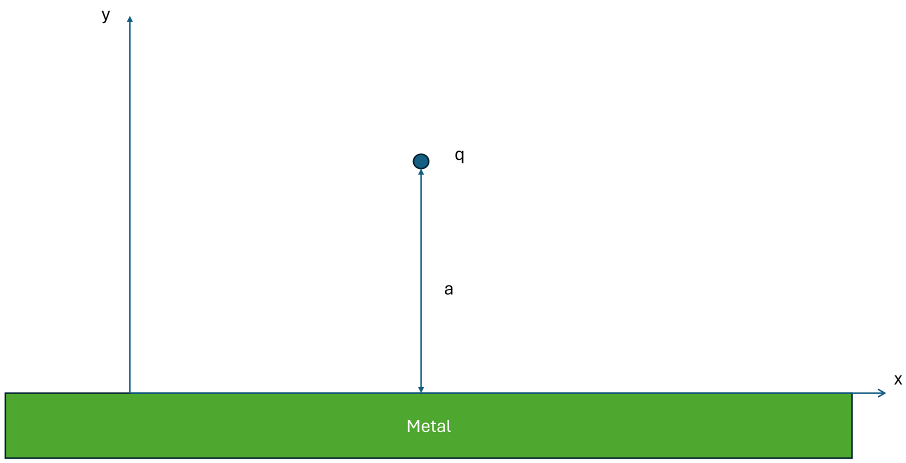

Figure 5: A point charge at $(0, a, 0)$ is above the metal plane at $y = 0$. The metal plane is infinite in the x and z directions. We show here only the projection in the x-y plane.

We compute the static electric field from the relation $\boldsymbol{E} = -\boldsymbol{\nabla} V$ and find

$$E_x = \frac{1}{4\pi\epsilon_0}\left(\frac{x}{(x^2+(y-a)^2+z^2)^{3/2}} - \frac{x}{(x^2+(y+a)^2+z^2)^{3/2}}\right), \tag{88}$$

$$E_y = \frac{1}{4\pi\epsilon_0}\left(\frac{y-a}{(x^2+(y-a)^2+z^2)^{3/2}} - \frac{y+a}{(x^2+(y+a)^2+z^2)^{3/2}}\right), \tag{89}$$

$$E_z = \frac{1}{4\pi\epsilon_0}\left(\frac{z}{(x^2+(y-a)^2+z^2)^{3/2}} - \frac{z}{(x^2+(y+a)^2+z^2)^{3/2}}\right). \tag{90}$$

c) An electron moves with constant velocity v in a circle of radius R so that its position at time t is

$$\boldsymbol{r}_e(t) = R\left(\cos\frac{vt}{R}\hat{\mathbf{x}} + \sin\frac{vt}{R}\hat{\mathbf{y}}\right), \tag{91}$$

where $\hat{\mathbf{x}}$ is a unit vector along the x direction and $\hat{\mathbf{y}}$ is a unit vector along the y direction.

What is the charge density $\rho(\boldsymbol{r}, t)$ and the charge current density $\boldsymbol{J}(\boldsymbol{r}, t)$? Demonstrate that there is charge conservation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0. \tag{92}$$

Solution

The charge density is

$$\rho(\boldsymbol{r}, t) = -e\delta(\boldsymbol{r} - \boldsymbol{r}_e(t)) \tag{93}$$

and the charge current density is

$$\boldsymbol{J} = -e\boldsymbol{v}_e(t)\delta(\boldsymbol{r} - \boldsymbol{r}_e(t)) \,, \tag{94}$$

where the velocity is $\boldsymbol{v}_e(t) = d\boldsymbol{r}_e(t)/dt$.

By computing the time derivative of the charge density, we find

$$\frac{\partial \rho}{\partial t} = e\boldsymbol{v}_e(t) \cdot \boldsymbol{\nabla}\delta(\boldsymbol{r} - \boldsymbol{r}_e(t)) \tag{95}$$

$$= -\boldsymbol{\nabla}\boldsymbol{J} \,, \tag{96}$$

which we should demonstrate.

d) In the Lorentz gauge, the time-dependent scalar potential V and vector potential $\boldsymbol{A}$ in vacuum are

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3R \frac{\rho(\boldsymbol{R}, t_r)}{|\boldsymbol{R} - \boldsymbol{r}|} \,, \tag{97}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int d^3R \frac{\boldsymbol{J}(\boldsymbol{R}, t_r)}{|\boldsymbol{R} - \boldsymbol{r}|} \,. \tag{98}$$

Explain what the retarded time t_r is and why we must use this time t_r in these equations.

Solution

In a vacuum, electromagnetic waves travel at the speed of light c. The retarded time $t_r = t - |\boldsymbol{R} - \boldsymbol{r}|/c$ is the time the signal started to propagate from position $\boldsymbol{R}$ to reach position $\boldsymbol{r}$ at time t.

A Maxwell‘s Equations

Maxwell‘s equation in vacuum for the electric field $\boldsymbol{E}$, the displacement field $\boldsymbol{D}$, the magnetic induction $\boldsymbol{B}$, and the magnetic field $\boldsymbol{H}$ are

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f \,, \tag{99}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \,, \tag{100}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \,, \tag{101}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial D}{\partial t} \,, \tag{102}$$

in terms of the free charge density ρ_f and the free charge current density $\boldsymbol{J}_f$.

B Constitute Relations

In linear and isotropic media, we have

$$\boldsymbol{D} = \epsilon \boldsymbol{E} \,, \tag{103}$$

$$\boldsymbol{B} = \mu H \,, \tag{104}$$

where ϵ is the dielectric constant and μ is the magnetic permeability.

C Boundary Conditions for Electromagnetic Fields

At interfaces between material 1 and material 2, the boundary conditions are

$$\hat{\mathbf{e}}_n \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0 \,, \tag{105}$$

$$\hat{\mathbf{e}}_n \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{K}_s \,, \tag{106}$$

$$\hat{\mathbf{e}}_n \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \sigma_S \,, \tag{107}$$

$$\hat{\mathbf{e}}_n \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0 \,, \tag{108}$$

where $\hat{\mathbf{e}}_n$ is a unit vector normal to the interface, σ_S is the surface charge density, and $\boldsymbol{K}_s$ is the surface charge current density.

D Spherical Coordinates

In spherical coordinates r, θ, and ϕ, the gradient is

$$\nabla t = \hat{\mathbf{r}} \frac{\partial t}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial t}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial t}{\partial \phi} \,. \tag{109}$$

The divergence is

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\,. \tag{110}$$

The Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,. \tag{111}$$

E Products of matrices

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})\ , \tag{112}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\,(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}\,(\boldsymbol{A}\cdot\boldsymbol{B})\ . \tag{113}$$

F Integral Theorems

The divergence theorem is

$$\int d^3r\nabla\cdot\boldsymbol{A} = \oint \boldsymbol{A}\cdot d\boldsymbol{S}\,. \tag{114}$$

Stoke‘s theorem (or the curl theorem) is

$$\int (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{S} = \oint \boldsymbol{A}\cdot d\boldsymbol{l}\,. \tag{115}$$

G Some Useful Results

We define $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_1$, $R = |\boldsymbol{R}|$, and $\hat{\mathbf{R}} = \boldsymbol{R}/R$. Then

$$\boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{R}}}{R^2} = 4\pi\delta(\boldsymbol{R})\,, \tag{116}$$

$$\boldsymbol{\nabla}\frac{1}{R} = -\frac{\hat{\mathbf{R}}}{R^2}\,, \tag{117}$$

$$\boldsymbol{\nabla}^2\frac{1}{R} = -4\pi\delta(\boldsymbol{R})\,. \tag{118}$$