Norwegian University of Science and Technology

Solution to Exam in Electromagnetic theory 2025

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Problem 1 - Short questions (20 points)

The sub-problems in this problem deal mostly with different systems and can be answered independently of each other. The questions should only require short derivations or short explanations.

A point dipole can be constructed in the following way: Consider two point charges, one of charge q and the other of charge $-q$, separated by some vector $\mathbf{d}$ (from $-q$ to q). The point dipole is obtained by letting $|\mathbf{d}| \to 0$, in such a way that the product $q|\mathbf{d}|$ is finite.

(a) Show that a point dipole situated at $\mathbf{r} = \mathbf{r}_0$ is described by the charge density

$$\rho(\mathbf{r}) = -\mathbf{p} \cdot \boldsymbol{\nabla}\delta(\mathbf{r} - \mathbf{r}_0), \tag{1}$$

where $\mathbf{p} := q\mathbf{d}$. **Hint**: Use a Taylor expansion.

Solution. The charge density of the point charges is $\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}_0) - q\delta(\mathbf{r} - \mathbf{r}_0 + \mathbf{d})$ [draw a figure to get the signs correct!]. When $\mathbf{d} \to 0$ we can Taylor expand the density to linear order in d (using the hint and the recipe above)

$$\begin{aligned}\rho(\mathbf{r}) &= q\delta(\mathbf{r} - \mathbf{r}_0) - q\delta(\mathbf{r} - \mathbf{r}_0) - q\mathbf{d} \cdot \boldsymbol{\nabla}\delta(\mathbf{r} - \mathbf{r}_0) + \ldots \\ &= -q\mathbf{d} \cdot \boldsymbol{\nabla}\delta(\mathbf{r} - \mathbf{r}_0) + \ldots \\ &\to -\mathbf{p} \cdot \boldsymbol{\nabla}\delta(\mathbf{r} - \mathbf{r}_0),\end{aligned}$$

where we take the limit $d \to 0$ while fixing $qd = p$ in the last line, which yields the advertised result.

(b) Use the charge density in Eq. (1) to derive the scalar potential $V(\mathbf{r})$ of the point dipole.

Solution. The scalar potential is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{2}$$

Inserting the charge density in Eq. (1) yields

$$\begin{aligned}V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3r' \frac{1}{|\mathbf{r} - \mathbf{r}'|}(-\mathbf{p} \cdot \boldsymbol{\nabla}')\delta(\mathbf{r}' - \mathbf{r}_0) \\ &= \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3r' \delta(\mathbf{r}' - \mathbf{r}_0)(+\mathbf{p} \cdot \boldsymbol{\nabla}')\frac{1}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3r' \delta(\mathbf{r}' - \mathbf{r}_0)\frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3},\end{aligned}$$

where we have performed integration by parts in passing from the first line, performed the gradient of $1/|\mathbf{r}-\mathbf{r}'|$ with respect to $\mathbf{r}'$ in the second and resolved the integral over the delta function in the last, fixing $\mathbf{r}' = \mathbf{r}_0$. We recognise this as the potential of a dipole with dipole moment $\mathbf{p}$.

In the absence of sources, the electromagnetic momentum density $\mathbf{g} := \mathbf{D} \times \mathbf{B}$ can be shown to satisfy the equation

$$\partial_t \mathbf{g} - \boldsymbol{\nabla} \cdot \overset{\leftrightarrow}{\mathbf{T}} = 0, \tag{3}$$

where $\overset{\leftrightarrow}{\mathbf{T}} := T_{ij} \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j$ denotes the Maxwell stress tensor.

(c) Explain the physical meaning of Eq. (3).

Solution. The above equation says that the change of momentum density per unit time at a point is equal to the flux of momentum density out of that point (i.e., the divergence). Hence, $-\overset{\leftrightarrow}{\mathbf{T}}$ is interpreted as the momentum current. This is the conservation law for momentum.

Suppose we describe a metal as a gas of mobile electrons flowing through the crystal lattice of the positive ions. On macroscopic scales, the metal is overall charge neutral. When we place a fixed charge Q into this system, say at $\mathbf{r}_0$, the electron charge density $\rho(\mathbf{r})$ of the metal is altered. A simple model for the scalar potential in the metal due to the charge Q is given by

$$\left(\Delta - \frac{1}{\lambda^2} \right) V(\mathbf{r}) = -\frac{Q}{\epsilon_0} \delta(\mathbf{r} - \mathbf{r}_0), \tag{4}$$

where λ is a constant called the *Debye screening length*, and $\Delta := \nabla^2$ is the Laplacian. You are given that the Green's function $G_{\hat{Y}}(\mathbf{r} - \mathbf{r}')$ of the operator $\hat{Y} := \Delta - k^2$ is

$$G_{\hat{Y}}(\mathbf{r} - \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \exp\left(-k|\mathbf{r} - \mathbf{r}'|\right). \tag{5}$$

We use the convention that $G_{\hat{Y}}$ satisfies $\hat{Y} G_{\hat{Y}}(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$.

(d) Write the solution to Eq. (4) using the Green's function. Give a brief physical interpretation of the solution.

Solution. The Green's function solves the general differential equation $\hat{Y} F(\mathbf{r}) = H(\mathbf{r})$ in the sense that

$$F(\mathbf{r}) = \int \mathrm{d}^3 r' G_{\hat{Y}}(\mathbf{r} - \mathbf{r}') H(\mathbf{r}'). \tag{6}$$

Using this with $F(\mathbf{r}) = V(\mathbf{r})$ and $H(\mathbf{r}) = -Q\delta(\mathbf{r} - \mathbf{r}_0)/\epsilon_0$ and the given Green's function yields

$$V(\mathbf{r}) = \frac{Q \mathrm{e}^{-|\mathbf{r} - \mathbf{r}_0|/\lambda}}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_0|}. \tag{7}$$

When placing a charge Q at a fixed position $\mathbf{r}_0$ in a metal, the free charges of opposite sign will be attracted to it (or free charges of the same sign will be repulsed from it), so that they partially cancel the potential from the charge Q. The result is a potential that when viewed from afar (i.e., at distances $\gtrsim \lambda$ from $\mathbf{r}_0$) looks like the same as that of a point charge in vacuum but with a reduced charge $\tilde{Q} = Q\mathrm{e}^{-|\mathbf{r}-\mathbf{r}_0|/\lambda}$. On macroscopic scales ($|\mathbf{r}-\mathbf{r}_0| \gtrsim \lambda$), we still have $V \simeq 0$ inside the metal, which is a known result in electrostatics.

Problem 2 - Radiation of a point charge (30 points)

In the Lorentz gauge, the scalar and vector potential satisfy the equations

$$\Box V = \rho/\epsilon_0 \quad \text{and} \quad \Box \mathbf{A} = \mu_0 \mathbf{J}, \tag{8}$$

where

$$\Box := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta, \tag{9}$$

is the wave operator. Recall that the retarded Green's function[1] of the wave operator is given by

$$G(\mathbf{r}-\mathbf{r}'; t-t') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta\left(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right). \tag{11}$$

Consider a particle with charge q moving along a trajectory $\mathbf{r}_q(t)$.

(a) What is $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ for the moving point charge?

Solution. The charge density is singular on the trajectory of the point particle. Moreover, $\mathbf{J}(\mathbf{r}, t) = \mathbf{v}_q(t)\rho(\mathbf{r}, t)$, so

$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{r}_q(t)) \quad \text{and} \quad \mathbf{J}(\mathbf{r}, t) = q\mathbf{v}_q(t)\delta(\mathbf{r} - \mathbf{r}_q(t)), \tag{12}$$

where $\mathbf{v}_q(t) = \dot{\mathbf{r}}_q(t)$.

Using the Green's function, we showed in the lectures that (you are not asked to derive this)

$$V(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R - \mathbf{R}\cdot\mathbf{v}_q/c}\right] \quad \text{and} \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi}\left[\frac{\mathbf{v}_q}{R - \mathbf{R}\cdot\mathbf{v}_q/c}\right], \tag{13}$$

where $\mathbf{R}(t) := \mathbf{r} - \mathbf{r}_q(t)$ and $\mathbf{v}_q(t) := \dot{\mathbf{r}}_q(t)$.

(b) What is the equation that determines the time at which the quantities in the square brackets of Eq. (13) are evaluated?

Why are these quantities *not* evaluated at time t?

Solution. The quantities in the square brackets are evaluated at $t_r = t - R(t_r)/c$, where $R(t) = |\mathbf{r} - \mathbf{r}_q(t)|$. They are not evaluated at time t because of the

[1] We have used the convention that G satisfies the equation

$$\Box G(\mathbf{r}-\mathbf{r}'; t-t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{10}$$

principle of causality: a signal arriving at some point $(t, \mathbf{r})$ cannot originate from a source outside the light cone centred at $(t, \mathbf{r})$.

If one does not remember the equation, it can be derived with the Green's function: Using the definition of a Green's function, we can write down the solution

$$\begin{aligned} V(\mathbf{r}, t) &= \int \mathrm{d}^3 r' \int \mathrm{d}t' G(\mathbf{r}-\mathbf{r}'; t-t') \frac{\rho(\mathbf{r}', t')}{\epsilon_0} \\ &= \frac{q}{4\pi\epsilon_0} \int \mathrm{d}t' \frac{1}{|\mathbf{r}-\mathbf{r}_q(t')|} \delta\left(t - t' - \frac{|\mathbf{r}-\mathbf{r}_q(t')|}{c} \right), \end{aligned} \tag{14}$$

where we have done the integral over $\mathbf{r}'$ directly. The remaining delta function fixes $t' = t_r$, where $t_r = t - R(t_r)/c$.

(c) Show that

$$V(\mathbf{r}, t) \simeq \frac{q}{4\pi\epsilon_0 r} (1 + \hat{\mathbf{r}} \cdot \dot{\mathbf{r}}_q(t - r/c)/c) \quad \text{and} \quad \mathbf{A}(\mathbf{r}, t) \simeq \frac{q\mu_0}{4\pi r} \dot{\mathbf{r}}_q(t - r/c), \tag{15}$$

in the far-field and non-relativistic limit. State how and where these approximations are used at each step in the derivation.

Hint: You may find the following relations useful:

$$\sqrt{1+x} \simeq 1 + \frac{x}{2} \quad \text{and} \quad \frac{1}{1+x} \simeq 1 - x \quad \text{for } x \ll 1. \tag{16}$$

Solution. To approximate the potentials, we should first specify precisely what these limits mean. The far-field (FF) limit means that $r \gg r_q(t)$, and the non-relativistic (NR) limit means that $v_q(t)/c \ll 1$, for all times. The equation for the retarded time can be approximated by

$$\begin{aligned} t_r &= t - \frac{|\mathbf{r} - \mathbf{r}_q(t_r)|}{c} \\ &= t - \frac{r}{c}\sqrt{1 - 2\hat{\mathbf{r}} \cdot \frac{\mathbf{r}_q(t_r)}{r} + \left(\frac{r_q(t_r)}{r}\right)^2} && \text{expanding length of vector} \\ &\simeq t - \frac{r}{c} + \hat{\mathbf{r}} \cdot \frac{\mathbf{r}_q(t_r)}{r}. && \text{Using FF, neglecting terms } \mathcal{O}((r_q/r)^2) \end{aligned}$$

The last term in the equation above is a correction to t given by $\hat{\mathbf{r}} \cdot \frac{\mathbf{r}_q(t_r)}{rt}$. If T is a typical time scale (e.g. the period of oscillatory motion) and L is a typical length scale of the motion of the particle, then this correction is $r_q(t_r)/(tc) \sim L/(Tc) \sim V/c$, i.e., a typical velocity divided by c. In the NR approximation we can neglect this, and so $t_r \simeq t - r/c$.

As for the factors of $1/(R - \mathbf{R}\cdot\mathbf{v}_q/c)$ we approximate the remaining appearances of $\mathbf{R}$ as simply $\mathbf{r}$ in the FF approximation. Since $\mathbf{A}$ is already of order $\dot{\mathbf{r}}/c$ we replace $1/(r - \mathbf{r}\cdot\mathbf{v}_q/c)$ by $1/r$ in the NR approximation. For V we keep the leading term in $\mathbf{v}_q/c$ by expanding

$$\frac{1}{r - \mathbf{r}\cdot\mathbf{v}_q/c} \simeq \frac{1}{r}\left(1 + \hat{\mathbf{r}}\cdot\mathbf{v}_q/c\right), \tag{17}$$

in the NR approximation. This yields the two advertised expressions.

Suppose that you are observing the moving point charge from a large distance $r \gg r_q(t)$ and you are interested in the electromagnetic *radiation* from the particle.

(d) Explain what is meant by *the radiation fields* of the particle.

Solution. The radiation fields are the terms of the electric and magnetic fields that fall off as $1/r$ at large distances. These contribute to the power flux density $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$ with a term that falls off as $1/r^2$, which compensates the scaling of the surface area of a sphere, meaning that the source radiates off to infinity.

(e) It is well known that accelerated charges emit electromagnetic radiation. Can this be qualitatively seen from the expressions in Eq. (15)? (You are not required to compute $\mathbf{E}$ and $\mathbf{B}$ explicitly.

Solution. From the expressions in Eq. (15) we see that $\boldsymbol{\nabla}V$ will get a contribution that is $\sim 1/r$ from the dependence on r in the argument of the velocity. Hence, by the chain rule, this is a contribution that is $\sim \mathbf{a}/r$. Likewise, $\partial_t\mathbf{A} \sim \mathbf{a}/r$. This shows that the electric field that falls off as $1/r$ is proportional to the acceleration of the particle. A similar argument can be made for $\mathbf{B}$. Hence, it takes accelerating charges to produce electromagnetic radiation.

Problem 3 - Cylindrical permanent magnet (30 points)

Consider a cylinder with radius a and length L. The cylinder has permanent and uniform magnetization M_0 parallel to its axis, and $\mathbf{J}_f = 0$ everywhere. That is,

$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & \mathbf{r} \in \text{cylinder} \\ 0 & \mathbf{r} \notin \text{cylinder} \end{cases}. \tag{18}$$

Place the coordinate system so that the cylinder extends from $-L/2 \leq z \leq L/2$, $0 \leq r \leq a$ and $0 \leq \phi < 2\pi$ (see Fig. 1).

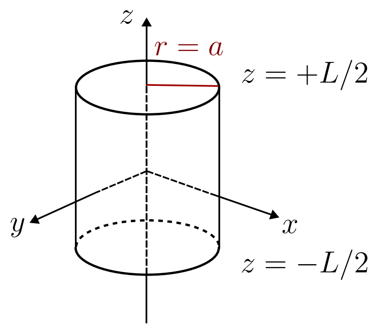

Figure 1: The figure shows the placement of the coordinate system with respect to the cylinder. The origin of the coordinate system lies at the centre of the cylinder.

(a) Argue that $\mathbf{H}$ in this case can be derived from a scalar potential $\varphi_M(\mathbf{r})$ such that $\mathbf{H}(\mathbf{r}) = -\boldsymbol{\nabla}\varphi_M(\mathbf{r})$. Show that $\varphi_M(\mathbf{r})$ satisfies the Poisson equation

$$\Delta\varphi_M = -\rho_M, \tag{19}$$

where $\rho_M(\mathbf{r}) = -\boldsymbol{\nabla} \cdot \mathbf{M}(\mathbf{r})$.

Solution. In a medium with $\mathbf{J}_f = 0$, the $\mathbf{H}$ field satisfies $\boldsymbol{\nabla} \times \mathbf{H} = 0$ ($\partial_t\mathbf{D} = 0$ in magnetostatics). This means that we can express it as the gradient of a scalar field $\mathbf{H} = -\boldsymbol{\nabla}\varphi_M$. Moreover, the equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ now translates to

$$\boldsymbol{\nabla} \cdot (\mathbf{H} + \mathbf{M}) = 0 \quad \Rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\varphi_M = \boldsymbol{\nabla} \cdot \mathbf{M}, \tag{20}$$

so

$$\Delta\varphi_M = -\rho_M, \tag{21}$$

with $\rho_M(\mathbf{r}) = -\boldsymbol{\nabla} \cdot \mathbf{M}(\mathbf{r})$.

At this point, it might be helpful to recall that the mathematical structure of the equations is exactly the same as those of electric polarization in the absence of free charge, which can be seen by doing the replacements

$$\mathbf{M}(\mathbf{r}) \longleftrightarrow \mathbf{P}(\mathbf{r}) \quad \text{and} \quad \varphi_M(\mathbf{r}) \longleftrightarrow \epsilon_0 V(\mathbf{r}). \tag{22}$$

(b) Show that $\varphi_M(\mathbf{r})$ can be determined from the expression

$$\varphi_M(\mathbf{r}) = \frac{1}{4\pi}\int_S \mathrm{d}a' \frac{\hat{\mathbf{n}}(\mathbf{r}') \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{23}$$

where S is the surface of the cylinder, and $\hat{\mathbf{n}}$ is its outward normal.

Hint: You can either make use of the analogy in Eq. (22) directly, or note that $\mathbf{M}$ in Eq. (18) is discontinuous across the surface of the cylinder. This means that its divergence $\boldsymbol{\nabla} \cdot \mathbf{M}$ is singular on the surface.

Solution. With the hint and the comment before problem (b), we can use our knowledge about electric polarization combined with the general solution to the Poisson equation to write down the potential. For polarization, the bound volume charge density is given by $\rho = -\boldsymbol{\nabla} \cdot \mathbf{P}$, while the bound surface charge density is $\sigma = \hat{\mathbf{n}} \cdot \mathbf{P}$. Hence, since $\mathbf{M}$ is constant inside the cylinder, only the surface charge will contribute here. The surface charge corresponding to the volume charge density $-\boldsymbol{\nabla} \cdot \mathbf{M}$ is given by $\mathbf{n} \cdot \mathbf{M}$, from which the result follows.

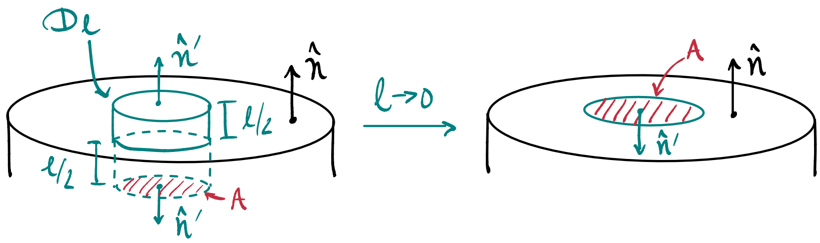

Figure 2: Picture of argument leading to the identification of the surface charge from the singular volume charge.

An alternative way to arrive at this result which does not rely on using the analogy to electrical polarization is the following. The density of "magnetic charge" $-\boldsymbol{\nabla} \cdot \mathbf{M}$ vanishes everywhere inside the cylinder since the magnetization is constant inside. On the surface, however, $\mathbf{M}$ is discontinuous so that its divergence is singular. We realize that since $\mathbf{M} \propto \hat{\mathbf{z}}$, the density is only singular on the circular discs with $\hat{\mathbf{n}} \propto \hat{\mathbf{z}}$, so let us focus on these surfaces. By choosing a small pillbox-shaped integration volume D_l that penetrates the surface and extends a short distance $l/2$ inside and outside the cylinder, we can use the divergence theorem to get

$$\int_{D_l} \mathrm{d}^3 r \rho_M(\mathbf{r}) = -\int_{D_l} \mathrm{d}^3 r \boldsymbol{\nabla} \cdot \mathbf{M}(\mathbf{r}) = \int_{\partial D_l} \mathrm{d}a \left(-\hat{\mathbf{n}}' \cdot \mathbf{M}(\mathbf{r})\right), \tag{24}$$

where $\hat{\mathbf{n}}'$ is the outward normal of D_l. The only finite contribution on the right-hand side comes from the surface of D_l that lies *inside* the magnet (where $\mathbf{M} \neq 0$), denoted by A. On this surface $\hat{\mathbf{n}}' = -\hat{\mathbf{n}}$. When $l \to 0$, D_l reduces to only the small disc A constituting the flat face of the pillbox (see figure 2).

Thus, we get only a contribution from the disc A that lies on the surface of the magnet on the left-hand side of the above equation, which is equal to

$$\int_A \mathrm{d}a \sigma_M(\mathbf{r}) = \int_A \mathrm{d}a \hat{\mathbf{n}} \cdot \mathbf{M}(\mathbf{r}). \tag{25}$$

Since the pillbox was arbitrary, we conclude that $\sigma_M(\mathbf{r}) = \hat{\mathbf{n}} \cdot \mathbf{M}(\mathbf{r})$ on the entire surface S, and the result follows by using

$$\varphi_M(\mathbf{r}) = \frac{1}{4\pi} \int_S \mathrm{d}a' \frac{\sigma_M(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{26}$$

(c) Compute $\varphi_M(z)$ and use it to determine the magnetic fields $\mathbf{H}$ and $\mathbf{B}$ along the z-axis, inside and outside the cylinder. That is, compute explicit expressions for $\mathbf{H}(z)$ and $\mathbf{B}(z)$ from $\varphi_M(z)$.

NB: you should **not** find the fields as functions of the polar angle ϕ and the radius r in the xy-plane.

Solution. We first compute $\varphi_M(z)$. Noticing that $\hat{\mathbf{n}} \cdot \mathbf{M}$ is non-zero only at the circular discs at $z = \pm L/2$, we realize that only these surfaces contribute to the surface integral in φ_M. Let us introduce some convenient notation: denote these discs by $\bigcirc_s$, with $s = \pm 1$ corresponding to $z = \pm L/2$. On these surfaces $\hat{\mathbf{n}} \cdot \mathbf{M} = s M_0$ since $\mathbf{n}$ is the outward normal. Hence,

$$\varphi_M(z) = \frac{M_0}{4\pi} \sum_{s=\pm 1} \int_{\bigcirc_s} \mathrm{d}a' \frac{s}{|\mathbf{r} - \mathbf{r}'|}. \tag{27}$$

Next, we compute $|\mathbf{r} - \mathbf{r}'|$ on the discs in cylindrical coordinates. From $\mathbf{r}' = r\hat{\mathbf{r}} + \hat{\mathbf{z}} sL/2$, we find

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + (z - sL/2)^2}. \tag{28}$$

Hence,

$$\begin{aligned} \varphi_M(z) &= \frac{M_0}{4\pi} \sum_{s=\pm 1} s \int_0^a \mathrm{d}r r \int_0^{2\pi} \mathrm{d}\phi \frac{1}{\sqrt{r^2 + (z - sL/2)^2}} \\ &= \frac{M_0}{2} \sum_{s=\pm 1} s \int_0^a \mathrm{d}r \frac{r}{\sqrt{r^2 + (z - sL/2)^2}}. \end{aligned} \tag{29}$$

The integral is solved with a substitution $u = (z - sL/2)^2 + r^2$ and $\mathrm{d}u = 2r\mathrm{d}r$, which yields

$$\begin{aligned} \varphi_M(z) &= \frac{M_0}{2} \sum_{s=\pm 1} s \int_{(z-sL/2)^2}^{(z-sL/2)^2+a^2} \frac{\mathrm{d}u}{2\sqrt{u}} \\ &= \frac{M_0}{2} \sum_{s=\pm 1} s \left(\sqrt{(z - sL/2)^2 + a^2} - |z - sL/2| \right) \\ &= \frac{M_0}{2} \left(\sqrt{(z - L/2)^2 + a^2} - \sqrt{(z + L/2)^2 + a^2} - |z - L/2| + |z + L/2| \right). \end{aligned}$$

Now, $\mathbf{H} = -\boldsymbol{\nabla}\varphi_M$. By symmetry, $\mathbf{H}$ must point along $\hat{\mathbf{z}}$ on the z axis. This means that $\mathbf{H}(z) = H_z(z)\hat{\mathbf{z}}$ and

$$\begin{aligned} H_z &= -\frac{\partial \varphi_M}{\partial z} \\ &= -\frac{M_0}{2}\left[\frac{z - L/2}{\sqrt{(z-L/2)^2 + a^2}} - \frac{z + L/2}{\sqrt{(z+L/2)^2 + a^2}} \right. \\ &\qquad \left. + \operatorname{sgn}(z + L/2) - \operatorname{sgn}(z - L/2)\right], \end{aligned} \tag{30}$$

where $\operatorname{sgn}(x) = x/|x|$. In total, we find

$$H_z(z) = \frac{M_0}{2}\begin{cases} \dfrac{z + L/2}{\sqrt{(z+L/2)^2 + a^2}} - \dfrac{z - L/2}{\sqrt{(z-L/2)^2 + a^2}} - 2 & |z| < L/2 \\ \dfrac{z + L/2}{\sqrt{(z+L/2)^2 + a^2}} - \dfrac{z - L/2}{\sqrt{(z-L/2)^2 + a^2}} & |z| > L/2 \end{cases}. \tag{31}$$

and since $B_z = \mu_0(H_z + M(z))$ we get

$$B_z(z) = \frac{\mu_0 M}{2}\left[\frac{z + L/2}{\sqrt{(z+L/2)^2 + a^2}} - \frac{z - L/2}{\sqrt{(z-L/2)^2 + a^2}}\right] \tag{32}$$

The z-components of the fields $B_z(z)$ and $H_z(z)$ are plotted in Fig. 3.

(d) Which of these (left or right) plots shows $B_z(z)$ and which shows $H_z(z)$? Comment on the difference between $\mathbf{H}$ and $\mathbf{B}$.

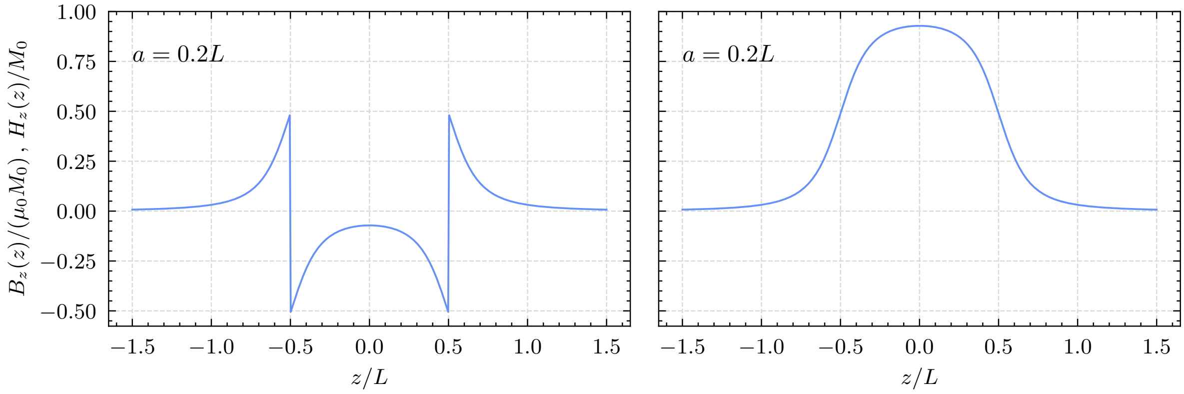

Figure 3: Plot of $H_z(z)$ and $B_z(z)$ with $a = 0.2L$.

Solution. The plot on the left-hand side shows H_z and the one on the right-hand side shows B_z. Note that the $\mathbf{B}$ field is continuous while $\mathbf{H}$ jumps at the surface of the cylinder. This can be traced back to the singular "magnetic charge density" of $\mathbf{H}$, which is $-\boldsymbol{\nabla} \cdot \mathbf{M}$. This charge density is however not physical, which is reflected by the absence of such discontinuity in $\mathbf{B}$. This is consistent with the boundary conditions of $\mathbf{B}$ and $\mathbf{H}$: the normal component of $\mathbf{B}$ is continuous across a boundary, but the normal component of $\mathbf{H}$ need not be.

Problem 4 - Gauge structure of electrodynamics (20 points)

In this course we have seen that the Maxwell theory has a *gauge structure.*

(a) What is a gauge transformation? Write down how the scalar potential $V(\mathbf{r}, t)$ and the vector potential $\mathbf{A}(\mathbf{r}, t)$ transform under such a transformation.

Solution. A gauge transformation is a transformation of the potentials that leave the physical fields unchanged. Specifically,

$$\mathbf{A} \mapsto \mathbf{A} + \boldsymbol{\nabla}\chi \quad \text{and} \quad V \mapsto V - \partial_t \chi, \tag{33}$$

where χ is an arbitrary (smooth) scalar function. This works because

$$\mathbf{E} = -\boldsymbol{\nabla} V - \partial_t \mathbf{A} \mapsto -\boldsymbol{\nabla} V - \partial_t \mathbf{A} + \boldsymbol{\nabla}\partial_t \chi - \partial_t \boldsymbol{\nabla}\chi = \mathbf{E}, \tag{34}$$

since partial derivatives commute, and

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \mapsto \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times \boldsymbol{\nabla}\chi = \mathbf{B}, \tag{35}$$

since the curl of a gradient vanishes (which is also due to partial derivatives commuting).

In class, we saw that two of the Maxwell equations in covariant form

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \tag{36}$$

followed from using the principle of least action with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}, \tag{37}$$

and the action

$$S[A] = \frac{1}{c} \int \mathrm{d}^3 r \int \mathrm{d}t \, (\mathcal{L} - A_\mu J^\mu) \, . \tag{38}$$

We are using the metric with signature $g = \mathrm{diag}(1, -1, -1, -1)$ and we have defined

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{39}$$

and

$$\partial_\mu := (\partial_t / c, \boldsymbol{\nabla}) \quad A^\mu := (V/c, \mathbf{A}) \quad \text{and} \quad J^\mu := (c\rho, \mathbf{J}). \tag{40}$$

(b) Show that $F_{\mu\nu}$ is gauge-invariant.

Solution. By using the gauge transformation $A_\mu \mapsto A_\mu - \partial_\mu \chi$, we find

$$\begin{aligned} F_{\mu\nu} &\mapsto \partial_\mu (A_\nu - \partial_\nu \chi) - \partial_\nu (A_\mu - \partial_\mu \chi) \\ &= F_{\mu\nu} - (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \chi \\ &= F_{\mu\nu} \qquad \text{partial derivatives commute.} \end{aligned} \tag{41}$$

This implies that $\mathcal{L}$ is a gauge-invariant quantity. However, the action shown in Eq. (38) does not look gauge-invariant, because of the term where A_μ appears as itself and not via the field tensor.

(c) Derive the condition that the current J^μ has to satisfy for $S[A]$ in Eq. (38) to be gauge-invariant. Give a physical interpretation of this condition.

Hint: You can ignore boundary terms.

Solution. Let us consider $\int \mathrm{d}^4 x A_\mu J^\mu$ under a gauge transformation:

$$\begin{aligned}\int \mathrm{d}^4 x A_\mu J^\mu &\mapsto \int \mathrm{d}^4 x (A_\mu - \partial_\mu \chi) J^\mu \\ &= \int \mathrm{d}^4 x A_\mu J^\mu - \int \mathrm{d}^4 x (\partial_\mu \chi) J^\mu \\ &= \int \mathrm{d}^4 x A_\mu J^\mu + \int \mathrm{d}^4 x \chi \partial_\mu J^\mu \qquad \text{Integration by parts,} \end{aligned} \tag{42}$$

where we have neglected a boundary term as suggested in the hint. Hence, for this term to be invariant under the gauge transformation, we must require that $\partial_\mu J^\mu = 0$. This is nothing but the conservation law of charge (the continuity equation).

For some special two-dimensional materials (two spatial dimensions and one temporal dimension), the electromagnetic response is drastically different, and the Lagrangian density is given by

$$\tilde{\mathcal{L}} = \frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{43}$$

In this equation $\epsilon^{\mu\nu\rho}$ is the Levi-Civita symbol, and μ, ν, ρ are indices that take values in $\{0, 1, 2\}$. Its definition here is entirely analogous to the Levi-Civita symbol in three spatial dimensions: it is cyclic and completely antisymmetric in exchanging any pair of indices, and $\epsilon^{012} = 1$. The coefficient $k/(4\pi)$ is a real number.

In two spatial dimensions, the electric field $\mathbf{E}$ is a two-component vector defined as usual, i.e., $\mathbf{E} = -\boldsymbol{\nabla} V - \partial_t \mathbf{A}$, while the magnetic field is a scalar $B = \partial_x A_y - \partial_y A_x$. In two spatial dimensions, the electromagnetic field tensor is still given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{44}$$

(d) Is $\tilde{\mathcal{L}}$ in Eq. (43) invariant under gauge transformations?

Hint: The following fact may be useful

$$\epsilon^{\mu\nu\rho} \partial_\nu A_\rho = \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho}. \tag{45}$$

Solution. No. For the same reason that $A_\mu J^\mu$ is not gauge-invariant (see comment before problem 4c), $\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$ is not gauge invariant.

This can be seen the following way. Note that the antisymmetry of the Levi-Civita symbol permits writing (or using hint)

$$\epsilon^{\mu\nu\rho}\partial_\nu A_\rho = \frac{1}{2}\epsilon^{\mu\nu\rho}\left(\partial_\nu A_\rho - \partial_\rho A_\nu\right) = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}. \tag{46}$$

By problem (b) this term is invariant. However, in $\tilde{\mathcal{L}}$ it appears contracted with A_μ, which is not gauge-invariant:

$$\tilde{\mathcal{L}} = \frac{k}{8\pi}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \mapsto \frac{k}{8\pi}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} - \frac{k}{8\pi}\epsilon^{\mu\nu\rho}(\partial_\mu \chi)F_{\nu\rho}. \tag{47}$$

The last term is not zero.

(e) Show that the action

$$\tilde{S}[A] := \frac{1}{c}\int_M \mathrm{d}^3x \tilde{\mathcal{L}}, \tag{48}$$

is invariant under gauge transformations provided that we can ignore boundary terms.

Solution. By using the results of problem (d) and the hint we find,

$$\tilde{S}[A] \mapsto \tilde{S}[A] + \frac{k}{8\pi}\frac{1}{c}\int_M \mathrm{d}^3x \epsilon^{\mu\nu\rho}\chi\partial_\mu F_{\nu\rho}, \tag{49}$$

where we have neglected the boundary term as suggested. The additional term is zero since (using hint of previous problem again),

$$\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} = 2\epsilon^{\mu\nu\rho}\partial_\mu\partial_\nu A_\rho = 0, \tag{50}$$

being a contraction of a symmetric and anti-symmetric tensor.

Comment: Note that this is analogous to why the term $A_\mu J^\mu$ breaks gauge invariance at the level of the Lagrangian density, but does not at the level of the action, provided that the current is conserved (problem 4c). Here, the conservation of the current $\tilde{J}_\mu$ defined by

$$A_\mu \tilde{J}^\mu = A_\mu \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho, \quad \text{that is} \quad \tilde{J}^\mu \equiv \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho, \tag{51}$$

is a consequence of partial derivatives commuting. That is,

$$\partial_\mu \tilde{J}^\mu = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\partial_\mu\partial_\nu A_\rho = \frac{k}{8\pi}\epsilon^{\mu\nu\rho}\left(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\right)A_\rho = 0. \tag{52}$$
