Problem 1

Exam 14.12.2020

a) $$m\dot{v} + kv = F_0 + R(t)$$

b) $$\gamma \equiv \frac{k}{m}$$

$$\dot{v} + \gamma v = \frac{F_0}{m} + \frac{R(t)}{m}$$

$$\frac{d}{dt}\left(e^{\gamma t} v(t)\right) = \frac{1}{m}\left(F_0 + R\right)e^{\gamma t}$$

$$e^{\gamma t} v(t) - v_0 = \frac{F_0}{m}\int_0^t dt' e^{\gamma t'} + \frac{1}{m}\int_0^t dt' R(t') e^{\gamma t'}$$

$$v(t) = v_0 e^{-\gamma t} + \frac{F_0}{m\gamma}\left(1 - e^{-\gamma t}\right) + \frac{e^{-\gamma t}}{m}\int_0^t dt' R(t') e^{\gamma t'}$$

c) $$\langle v(t)\rangle = v_0 e^{-\gamma t} + \frac{F_0}{m\gamma}\left(1 - e^{-\gamma t}\right)$$

$$\langle v^2(t)\rangle = \langle v(t)\rangle^2 + \frac{e^{-2\gamma t}}{m^2}\int_0^t dt_1 \int_0^t dt_2 e^{\gamma(t_1+t_2)} \langle R(t_1)R(t_2)\rangle$$

$$\langle R(t_1) R(t_2) \rangle = \langle R^2 \rangle \, \delta(t_1 - t_2)$$ ②

Thus, the last integral becomes

$$I = \frac{e^{-2\gamma t}}{m^2} \langle R^2 \rangle \underbrace{\int_0^t dt' \, e^{2\gamma t'}}_{= \frac{1}{2\gamma}\left(e^{2\gamma t} - 1\right)}$$

Hence

$$\sigma^2 = \langle (v - \langle v \rangle)^2 \rangle = I$$

$$= \frac{\langle R^2 \rangle}{2\gamma m^2} \left(1 - e^{-2\gamma t}\right)$$

d) $t \to \infty$: Equilibrium

$$\frac{1}{2} m \sigma^2 = \frac{1}{2} k_B T$$ (Equipartition thm.)

$$\sigma^2 = \frac{\langle R^2 \rangle}{2\gamma m^2} = \frac{k_B T}{m}$$

$$\langle R^2 \rangle = 2\gamma m k_B T$$

~~d)~~ (e) $\langle v(t_1) v(t_2) \rangle$

$$= \langle v(t_1) \rangle \langle v(t_2) \rangle$$

$$+ \frac{e^{-\gamma(t_1 + t_2)}}{m^2} \int_0^{t_1} dt' \int_0^{t_2} dt'' \, e^{\gamma(t' + t'')} \langle R^2 \rangle \, \delta(t_1 - t_2)$$

Consider the final integral, I: (3)

i) $t_1 > t_2$:

$$I = \frac{e^{-\gamma(t_1+t_2)}}{m^2}\langle R^2\rangle \int_0^{t_2} dt'\, e^{2\gamma t'}$$

$$= \frac{\langle R^2\rangle}{2\gamma m^2}\left[e^{-\gamma(t_1-t_2)} - e^{-\gamma(t_1+t_2)}\right]$$

i) $t_2 > t_2$:

$$I = \frac{e^{-\gamma(t_1+t_2)}}{m^2}\langle R^2\rangle \int_0^{t_1} dt'\, e^{2\gamma t'}$$

$$= \frac{\langle R^2\rangle}{2\gamma m^2}\left[e^{-\gamma(t_2-t_1)} - e^{-\gamma(t_1+t_2)}\right]$$

$t_1 \to \infty$, $t_2 \to \infty$, $t_1 - t_2 = \tau$

$$\langle v(t_1)\rangle \to \frac{F_0}{m\gamma}$$

$$\langle v(t_2)\rangle \to \frac{F_0}{m\gamma}$$

$$I \to \frac{\langle R^2\rangle}{2\gamma m^2} e^{-\gamma|\tau|}$$

$$\langle v(t_1)v(t_2)\rangle \to \left(\frac{F_0}{m\gamma}\right)^2 + \frac{\langle R^2\rangle}{2\gamma m^2} e^{-\gamma|\tau|}$$

f) $\langle\langle v(t_1)\, v(t_2)\rangle\rangle$

= autocorrelation function

$= \langle v(t_1)\, v(t_2)\rangle - \langle v(t_1)\rangle\langle v(t_2)\rangle$

$$= \frac{\langle R^2\rangle}{2m^2\gamma}\left[e^{-\gamma|\tau|} - e^{-\gamma(t_1+t_2)}\right]$$

$t_1, t_2 \to \infty$: Exponentially decaying with $|\tau|$; short-time correlations. $v(t)$ is thus a Markov-process.

Note the importance of considering $\langle\langle v(t_1)\, v(t_2)\rangle\rangle$ instead of $\langle v(t_1) v(t_2)\rangle$ to address the question.

Alternatively, one may argue that $v(t)$ is determined uniquely by one initial condition.

Problem 2.

a) $$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{a} \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll.}$$

L.H.S: A convective (total) derivative in $(t, \vec{r}, \vec{v})$-space.

R.H.S: A collision-term.

$\vec{v}$: Velocity

$\vec{a}$: Accelleration

$\vec{r}$: Rosition

t: time

f: Single-particle distribution-function

$$f = f(\vec{r}, \vec{v}, t)$$

$\left(\frac{\partial f}{\partial t}\right)_{coll}$: Term describing collisions between pairs of particles.

b) When specifying the collision-term $\left(\frac{\partial f}{\partial t}\right)_{coll}$ in terms of a product of two f's for a pair of particles, a crucial assumption is that the colliding particles are statistically independent prior to (but not after) collision. This crucial assumption clearly breaks time-reversal symmetry.

c) H-theorem: Define a functional

$$H = \int d^3r \int d^3v \, f \ln f$$

Then $$\frac{dH}{dt} \leq 0$$

Thus, the Boltzmann-equation describes a temporal evolution of a system from an initial state towards an equilibrium state.

d) Boltzmann-equation with $\vec{a} = 0$ and $f = f(t)$

L.H.S: $\frac{\partial f}{\partial t}$

R.H.S: $a(f - f_0)$

f_0: Equilibrium solution

$\frac{\partial f_0}{\partial t} = 0$ (clearly)

$$\frac{\partial (f - f_0)}{\partial t} = a(f - f_0)$$

$$f = f_0 + \tilde{f_0}\, e^{at}$$

$a < 0$ for $f \to f_0$ $(t \to \infty)$

$a = -1/\tau$

τ: Relaxation time

$$\underline{\underline{f = f_0 + \tilde{f_0}\, e^{-t/\tau}}}$$

f_0: $\lim_{t \to \infty} f(t)$

$\underline{\tilde{f_0} = f(0) - f_0}$