



NTNU – Trondheim
Norwegian University of
Science and Technology

Examination paper for TFY4280 Signal Processing

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Examination date: 06.06.2014

Examination time (from-to): 0900 - 1300

Permitted examination support material:

- Simple calculator (according to NTNU exam regulations)
- K. Rottmann: Matematisk formelsamling (eller tilsvarende)
- Barnett and Cronin: Mathematical formulae
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler (eller tilsvarende)

Other information:

- Languages: English/Bokmål/Nynorsk
- Number of pages (including this page and attachments): 12
- Answer must be written in English or Norwegian. Number of points given to each sub-question is given in bold font. The maximum score for the exam is **100p**.
- **Attachment:** 2 pages with transform tables and properties.

Checked by:

Date:

Signature:

Q1: Impulse response (30p)

- A) (10p) Explain what is described by the concept of *impulse response*. For continuous-time or discrete-time LTI systems, how can *impulse response* be used to determine the output signal $y(t)$ from an arbitrary input signal $x(t)$. Consider both time and frequency domains. Are there any necessary assumptions? Explain.
- B) (10p) Find impulse response function for LTI system described by differential equation given below.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \quad (1)$$

- C) (10p) Find impulse response function for a system described by a difference equation given below.

$$9y[n] - 9y[n-1] + 2y[n-2] = x[n] - 2x[n-1] \quad (2)$$

Q2: Frequency response (20p)

- A) (10p) What is described by “frequency response” $H(j\omega)$ of a LTI system? Are initial conditions important to determine frequency response?
- B) (10p) Determine expression describing $H(j\omega)$ for system described by the differential equation given in question 1B. Calculate for $\omega = 0\text{Hz}$ and 10Hz .

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \quad (3)$$

- Q3 (30p)** Consider a periodic square wave signal $x(t)$ shown in Figure 1, assume that this function is defined for $-\infty < t < \infty$

- A) (10p) Derive a general expression for Fourier transform of a periodic function $x_p(t)$ and Fourier transform of a sampled function $x_s(t)$ (defined in discrete time domain).

HINT: These equations should be helpful :

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \quad (4)$$

$$\mathcal{F}\{\delta_T(t)\} = \sum_{n=-\infty}^{\infty} \omega_0 \delta(\omega - n\omega_0) \quad (5)$$

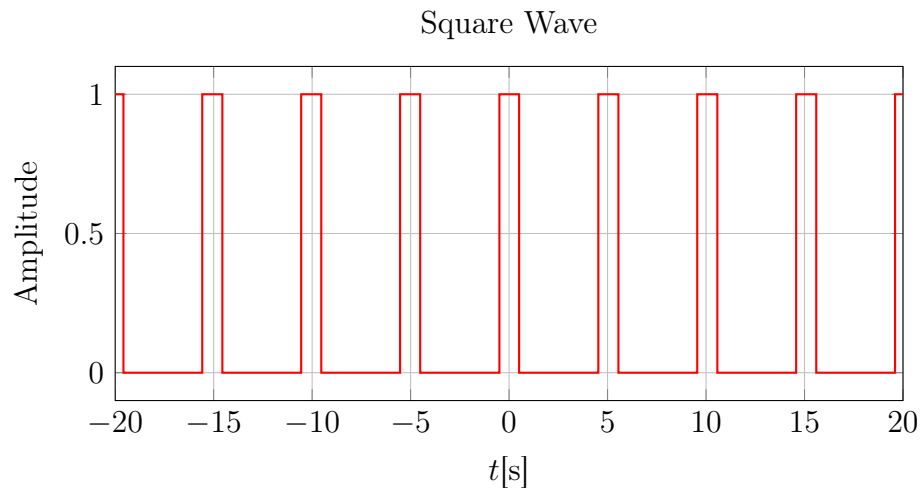


Figure 1: Square wave with repeat period of 5s and 1s pulse duration. Question 3.

- B) (10p)** Find the expression for the Fourier transform of the signal shown in Figure 1 and sketch it for $-2\pi < \omega < 2\pi$.
- C) (10p)** Define *power density spectrum* and explain how to calculate it for signal shown in Figure 1.
HINT: These equations might be helpful:

$$\Lambda\left[\frac{t}{T}\right] = \begin{cases} 1 - \frac{|t|}{T} & \text{for } |t| < T \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}\left\{\Lambda\left[\frac{t}{T}\right]\right\} = \frac{\sin^2(\omega T/2)}{(\omega T/2)^2}$$

Q4 (20p) Consider a system with impulse response $h(t) = e^t \epsilon(t)$ ($t \geq 0$) where $\epsilon(t)$ is the unit step function.

- A) (10p)** Is the system BIBO stable?
- B) (10p)** Now put this system into the feed-back system shown in Figure 2 and find the range of A-values so that the system is BIBO stable.

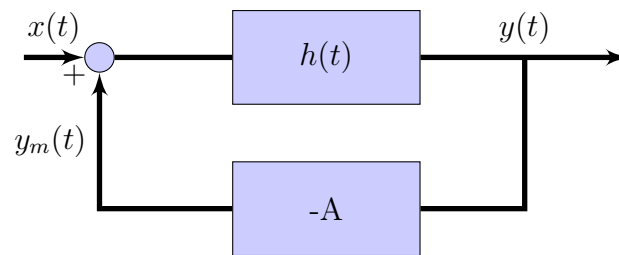


Figure 2: Feed-back system for Question 4B

Q1: Impulsrespons (30p)

A) (10p) Forklar hva som er beskrevet av konseptet *impulsrespons*. For kontinuerlige eller tidsdiskrete LTI-systemer, hvordan kan en impulsrespons brukes til å bestemme utgangssignalet $y(t)$ fra et vilkårlig inngangssignal $x(t)$. Betrakt både tids- og frekvensdomener. Er det noen nødvendige antakelser? Forklar.

B) (10p) Finn impulsresponsen for LTI-systemet beskrevet av differensialligning gitt nedenfor.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \quad (6)$$

C) (10p) Finn impulsresponsfunksjonen for et system som beskrives av differanseligningen gitt nedenfor.

$$9y[n] - 9y[n-1] + 2y[n-2] = x[n] - 2x[n-1] \quad (7)$$

Q2: Frekvensrespons (20p)

A) (10p) Hva er beskrevet av “frekvensrespons” $H(j\omega)$ for et LTI-system? Er initialtilstanden (“initial conditions”) viktig for å bestemme frekvensrespons?

B) (10p) Bestem uttrykket som beskriver $H(j\omega)$ for systemet beskrevet av differensialligningen gitt i spørsmålet 1B. Beregn for $\omega = 0$ Hz og 10 Hz.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \quad (8)$$

Q3 (30p) Betrakt et periodisk firkantbølgesignal $x(t)$ vist i figur 3. Anta at denne funksjonen er definert for $-\infty < t < \infty$

A) (10p) Utled et generelt uttrykk for Fouriertransformen av en periodisk funksjon $x_p(t)$ og Fouriertransformen av en samplet funksjon $x_s(t)$ (funksjon som er definert i diskret tidsdomene).

Tips: Disse ligningene bør være nyttige:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \quad (9)$$

$$\mathcal{F}\{\delta_T(t)\} = \sum_{n=-\infty}^{\infty} \omega_0 \delta(\omega - n\omega_0) \quad (10)$$

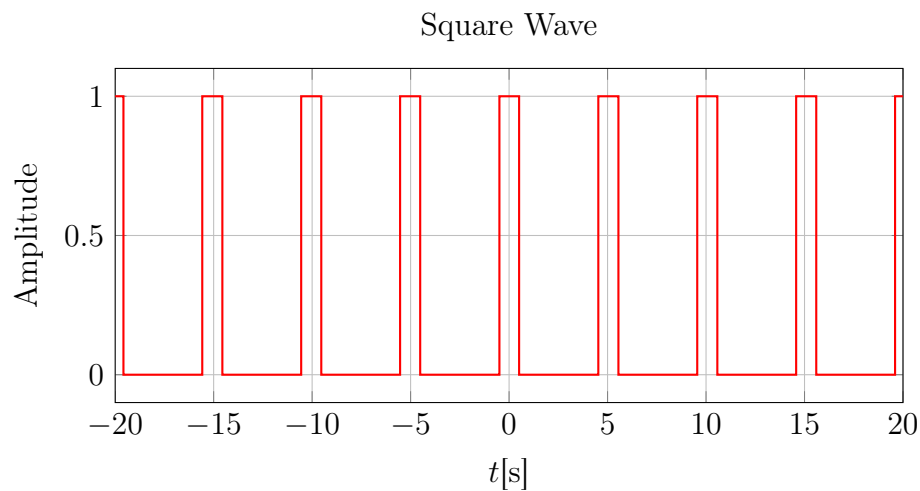


Figure 3: Firkantbølge med gjentakelsesperiode på 5s og 1s pulsvarighet. Spørsmål 3.

- B) (10p)** Finn uttrykket for Fouriertransformen av signalet vist i Figur 3 og skisser det for $-2\pi < \omega < 2\pi$.
- C) (10p)** Definer *effekttetthessspektrumet* (*power density spectrum*) og forklar hvordan man regner det for signalet vist i figur 3.
- Tips:** Disse ligningene kan være nyttige:

$$\Lambda\left[\frac{t}{T}\right] = \begin{cases} 1 - \frac{|t|}{T} & \text{for } |t| < T \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}\left\{\Lambda\left[\frac{t}{T}\right]\right\} = \frac{\sin^2(\omega T/2)}{(\omega T/2)^2}$$

Q4 (20p) Betrakt et system med impulsresponsen $h(t) = e^t \epsilon(t)$ ($t \geq 0$), hvor $\epsilon(t)$ er “unit step function”.

- A) (10p)** Er systemet BIBO stabilt?
- B) (10p)** Sett dette systemet i feed-back systemet vist i figur 4 og finn uttrykket for A , slik at systemet er BIBO stabilt.

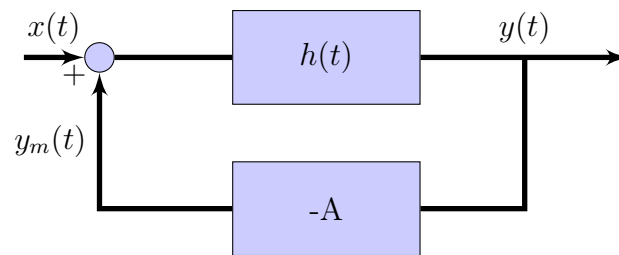


Figure 4: Feed-back system for Spørsmål 4B

Q1: Impulsrespons (30p)

A) (10p) Forklar kva som er beskreve av konseptet *impulsrespons*. For kontinuerlege eller tidsdiskrete LTI-systemer, korleis kan em impulsrespons brukas til å fastsetje utgangssignalet $y(t)$ frå eit vilkårlig inngangssignal $x(t)$. Betrakt både tids- og frekvensdomene. Er det nokre nødvendige føresetnader? Forklar.

B) (10p) Finn impulsrespons for eit LTI-system beskrive av differensiallikninga gitt under.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \quad (11)$$

C) (10p) Finn impulsresponsfunksjonen for eit system som beskrivas av differanselikninga gitt under.

$$9y[n] - 9y[n-1] + 2y[n-2] = x[n] - 2x[n-1] \quad (12)$$

Q2: Frekvensrespons (20p)

A) (10p) Kva er beskrevet av "frekvensrespons" $H(j\omega)$ for eit LTI-system? Er "initialtilstanden" ("initial conditions") viktig for å bestemme frekvensrespons?

B) (10p) Bestem uttrykket som beskriv $H(j\omega)$ for systemet beskrevet av differensiallikninga gitt i spørsmål 1B. Berekn for $\omega = 0$ Hz og 10 Hz.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \quad (13)$$

Q3 (30p) Betrakt det periodiske firkantbølgesignalet $x(t)$ vist i figur 5. Anta at denne funksjonen er definert for $-\infty < t < \infty$

A) (10p) Utled eit generelt uttrykk for Fouriertransformen av ein periodisk funksjon $x_p(t)$ og Fouriertransformen av ein samplet funksjon $x_s(t)$ (funksjon som er definert i diskret tidsdomene).

Tips: Disse likningane bør vere nyttige:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \quad (14)$$

$$\mathcal{F}\{\delta_T(t)\} = \sum_{n=-\infty}^{\infty} \omega_0 \delta(\omega - n\omega_0) \quad (15)$$

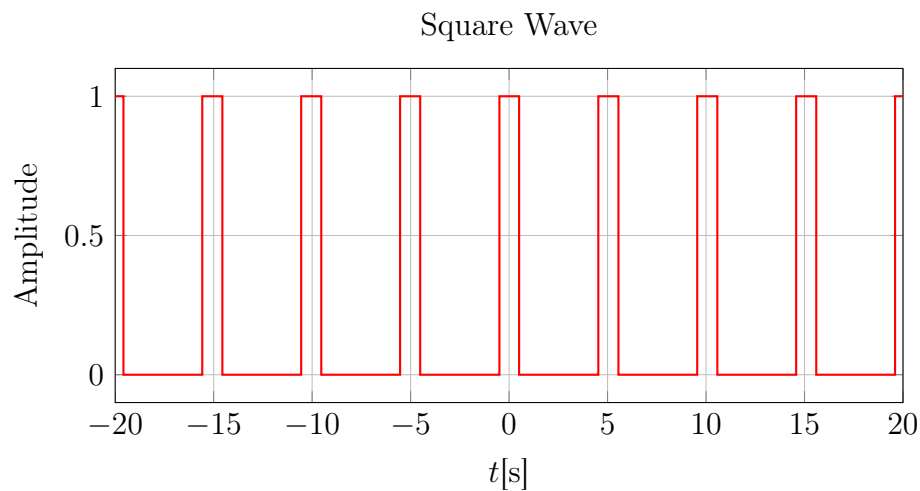


Figure 5: Firkantbølge med gjentakingsperiode på 5s og 1s pulstid. Spørsmål 3.

- B) (10p)** Finn uttrykket for Fouriertransformen av signalet vist i Figur 5 og skisser det for $-2\pi < \omega < 2\pi$.
- C) (10p)** Definer *power density spectrum* og forklar korleis ein reknar det ut for signalet vist i figur 5.
- Tips:** Disse likningane kan vere nyttige:

$$\Lambda\left[\frac{t}{T}\right] = \begin{cases} 1 - \frac{|t|}{T} & \text{for } |t| < T \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}\left\{\Lambda\left[\frac{t}{T}\right]\right\} = \frac{\sin^2(\omega T/2)}{(\omega T/2)^2}$$

Q4 (20p) Betrakt eit system med impulsresponsen $h(t) = e^t \epsilon(t)$ ($t \geq 0$), kor $\epsilon(t)$ er “unit step function”.

- A) (10p)** Er systemet BIBO stabilt?
- B) (10p)** Sett dette systemet i feed-back systemet vist i figur 6 og finn uttrykket for A , slik at systemet er BIBO stabilt.

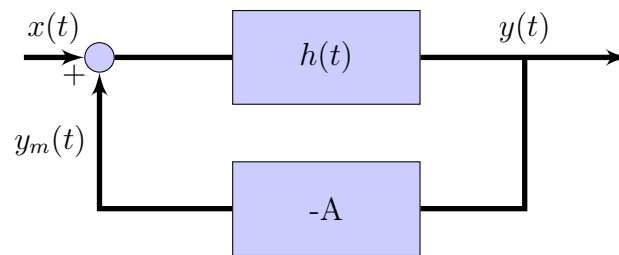


Figure 6: Feed-back system for Spørsmål 4B.

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\text{Re}\{s\} > 0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} < \text{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\text{Re}\{s\} > 0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}\{s\} > 0$
$t e^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$t^n e^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\text{Re}\{s\} > 0$
$\cos(\omega_0 t)\varepsilon(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\text{Re}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$e^{-at}\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$t\cos(\omega_0 t)\varepsilon(t)$	$\frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2}$	$\text{Re}\{s\} > 0$
$t\sin(\omega_0 t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2 + \omega_0^2)^2}$	$\text{Re}\{s\} > 0$

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\delta(t)$	$j\omega$
$\frac{1}{T} \text{III}\left(\frac{t}{T}\right)$	$\text{III}\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\text{rect}(at)$	$\frac{1}{ a } \text{si}\left(\frac{\omega}{2a}\right)$
$\text{si}(at)$	$\frac{\pi}{ a } \text{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi \text{sign}(\omega)$
$\text{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
$e^{-\alpha t }, \alpha > 0$	$\frac{2\alpha}{\alpha^2 + \omega^2}$
$e^{-a^2 t^2}$	$\frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}}$

Appendix B.2 Properties of the Bilateral Laplace Transform

$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
Linearity $Ax_1(t) + Bx_2(t)$	$AX_1(s) + BX_2(s)$	$\text{ROC} \supseteq \text{ROC}\{X_1\} \cap \text{ROC}\{X_2\}$
Delay $x(t - \tau)$	$e^{-s\tau}X(s)$	not affected
Modulation $e^{at}x(t)$	$X(s - a)$	$\text{Re}\{a\}$ shifted by $\text{Re}\{a\}$ to the right
'Multiplication by t ', Differentiation in the frequency domain $tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain $\frac{d}{dt}x(t)$	$sX(s)$	$\text{ROC} \supseteq \text{ROC}\{X\}$
Integration $\int_{-\infty}^t x(\tau)d\tau$	$\frac{1}{s}X(s)$	$\text{ROC} \supseteq \text{ROC}\{X\} \cap \{s : \text{Re}\{s\} > 0\}$
Scaling $x(at)$	$\frac{1}{ a }X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$
Delay	$x(t - \tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
'Multiplication by t ', Differentiation in the frequency domain $tx(t)$	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain $\frac{dx(t)}{dt}$	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^t x(\tau)d\tau$	$X(j\omega) \left[\pi\delta(\omega) + \frac{1}{j\omega} \right]$ $= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{ a }X\left(\frac{j\omega}{a}\right), \quad a \in \mathbb{R} \setminus \{0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega) \cdot X_2(j\omega)$
Multiplication	$x_1(t) \cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty} x(t) ^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) ^2 d\omega$

Appendix B.6 Properties of the z -Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k] + bx_2[k]$	$aX_1(z) + bX_2(z)$	$\text{ROC} \supseteq \text{ROC}\{X_1\} \cap \text{ROC}\{X_2\}$
Delay	$x[k - \kappa]$	$z^{-\kappa}X(z)$	$\text{ROC}\{x\}$; separate consideration of $z = 0$ and $z \rightarrow \infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	$\text{ROC} = \left\{z \mid \frac{z}{a} \in \text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z \frac{dX(z)}{dz}$	$\text{ROC}\{x\}$; separate consideration of $z = 0$
Time inversion	$x[-k]$	$X(z^{-1})$	$\text{ROC} = \{z \mid z^{-1} \in \text{ROC}\{x\}\}$
Convolution	$x_1[k] * x_2[k]$	$X_1(z) \cdot X_2(z)$	$\text{ROC} \supseteq \text{ROC}\{x_1\} \cap \text{ROC}\{x_2\}$
Multiplication	$x_1[k] \cdot x_2[k]$	$\frac{1}{2\pi j} \oint X_1(\zeta) X_2\left(\frac{z}{\zeta}\right) \frac{1}{\zeta} d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z -Transform Pairs

$x[k]$	$X(z) = \mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z \in \mathbf{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$ z > 1$
$a^k \varepsilon[k]$	$\frac{z}{z-a}$	$ z > a $
$-a^k \varepsilon[-k-1]$	$\frac{z}{z-a}$	$ z < a $
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$ z > 1$
$ka^k \varepsilon[k]$	$\frac{az}{(z-a)^2}$	$ z > a $
$\sin(\Omega_0 k) \varepsilon[k]$	$\frac{z \sin \Omega_0}{z^2 - 2z \cos \Omega_0 + 1}$	$ z > 1$
$\cos(\Omega_0 k) \varepsilon[k]$	$\frac{z(z - \cos \Omega_0)}{z^2 - 2z \cos \Omega_0 + 1}$	$ z > 1$