Norwegian University of Science and Technology
Department of Physics

Contacts during the exam:
Pawel Sikorski, phone: 98486426

EXAM
TFY4280 Signal Processing

Friday. 24th May 2013. 09:00

Examination support materials:

- Simple calculator (according to NTNU exam regulations)
- K. Rottmann: Matematisk formelsamling (eller tilsvarende)
- Barnett and Cronin: Mathematical formulae
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler (eller tilsvarende)

Answer must be written in English or Norwegian. Number of points given to each sub-question is given in bold font. The maximum score for the exam is **100p**. The exam consists of 4 questions. **Attachment:** 2 pages with transform tables and properties.

Q1 (25p) For a LTI system with unknown characteristics, a signal $x(t) = \varepsilon(t-t_0) - \varepsilon(t-t_1)$ results in a output $y(t)$ given by:

$$y(t) = h(t) * x(t) = \varepsilon(t-t_1)\left(e^{-(t-t_1)/2} - 1\right) - \varepsilon(t-t_0)\left(e^{-(t-t_0)/2} - 1\right)$$

A) (10p) Find the impulse response function $h(t)$ for this system.

To get the inpulse response one need first to find the Laplace thansform of the input and the output. We make use of time shift properties

$$y(t) = \varepsilon(t-t_1)\left(e^{-(t-t_1)/2} - 1\right) - \varepsilon(t-t_0)\left(e^{-(t-t_0)/2} - 1\right)$$

$$\begin{aligned}
Y(s) &= e^{-st_1}\left(\mathscr{L}\left\{\varepsilon(t)e^{-(t)/2}\right\} - \frac{1}{s}\right) + e^{-st_1}\left(\frac{1}{s} - \mathscr{L}\left\{\varepsilon(t)e^{-(t)/2}\right\}\right) = \\
&= e^{-st_1}\left(\frac{1}{s+0.5} - \frac{1}{s}\right) + e^{-st_0}\left(\frac{1}{s} - \frac{1}{s+0.5}\right) = \\
&= \left(e^{-st_1} - e^{-st_0}\right)\left(\frac{1}{s+0.5} - \frac{1}{s}\right) = \left(e^{-st_1} - e^{-st_0}\right)\left(\frac{s-s-0.5}{s(s+0.5)}\right) \\
&= \left(e^{-st_0} - e^{-st_1}\right)\left(\frac{0.5}{s(s+0.5)}\right) = \frac{1}{s}\left(e^{-st_0} - e^{-st_1}\right)\left(\frac{1}{1+s/0.5}\right)
\end{aligned}$$

The input

$$\begin{aligned}
\mathscr{L}\left\{\varepsilon(t-t_0) - \varepsilon(t-t_1)\right\} &= \left(e^{-st_0}\frac{1}{s} - e^{-st_1}\frac{1}{s}\right) = \\
&= \left(e^{-st_0} - e^{-st_1}\right)\frac{1}{s}
\end{aligned}$$

Comparing input with the output we see that:

$$H(s) = \frac{1}{1 + s/0.5} = \frac{0.5}{s + 0.5} =$$
$$h(t) = 0.5\varepsilon(t)e^{-0.5t}$$

B) (10p) Find the frequency response $|H(j\omega)|$ for $\omega = 1, 10, 100\text{Hz}$

From above

$$H(j\omega) = \frac{1}{1 + j\omega/0.5}$$
$$|H(j\omega)| = \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}}$$
$$\omega_c = 0.5$$

$$|H(1)| = 0.4472 = -7dB \quad |H(10)| = 0.0499 = -26dB \quad |H(100)| = 0.0050 = -46dB$$

C) (5p) Explain what is usually called by a term "white noise"? Explain how such noise signal $n(t)$ could be used to obtain $h(t)$ for LTI system with unknown characteristics.

White noise is defined by a random signal for with constant power spectral density (the noise contain the same amount of energy for a given frequency range, for all possible frequencies). This is a idealised concept. $\varphi_{nn} = \delta(t)$ and $\Phi_{nn}(j\omega) = N_0$. Note: stationary random signals can not be integrated absolutely, so in general we are not able to calculate FT for this type of signal (if $|t| \to \infty$). Instead we use FT of the expected value φ.

$$\begin{aligned}
y_0(t) &= n(t) * h(t) \\
y_1(t) &= \varphi_{yn}(\tau) = y_0(t) * n(-t) \\
y_1(t) &= n(t) * h(t) * n(-t) = n(t) * n(-t) * h(t) = \\
&= \varphi_{nn}(\tau) * h(t) = \delta(t) * h(t) = h(t)
\end{aligned}$$

So, since $n(t) * n(-t)$ is an autocorrelation of random noise and it is approaching $\delta(t)$, we can measure impulse response. This can also be explained using FT of autocorrelation function (power density spectrum $\Phi_{nn}(j\omega)$).

Q2 (25p) Consider the following difference equation and excitation $x[n]$ (input signal):

$$y[n] - 0.7y[n-1] + 0.1y[n-2] = x[n] + x[n-1]$$

$$x[n] = \begin{cases} 1 & n = 2 \\ 0 & n \neq 2 \end{cases}$$

A) (10p) Find y[n] using z-transform.

$$Y(z) - 0.7Y(z)z^{-1} + Y(z)0.1^{z-2} = X(z) + X(z)z^{-1}$$

$$\frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - 0.7z^{-1} + 0.1z^{-2}} = \frac{z^2 + z}{z^2 - 0.7z + 0.1} = \frac{z^2 + z}{(z - 0.5)(z - 0.2)}$$

$$\frac{H(z)}{z} = \frac{z + 1}{(z - 0.5)(z - 0.2)} = \frac{k_1}{(z - 0.5)} + \frac{k_2}{(z - 0.2)}$$

$$k_1 = \frac{0.5 + 1}{0.5 - 0.2} = 5 \quad k_2 = \frac{0.2 + 1}{0.2 - 0.5} = -4$$

check:

$$\frac{5}{(z - 0.5)} - \frac{4}{(z - 0.2)} = \frac{5z - 1 - 4z + 2}{(z - 0.5)(z - 0.2)} = \frac{z + 1}{(z - 0.5)(z - 0.2)}$$

So:

$$H(z) = \frac{5z}{(z-0.5)} - \frac{4z}{(z-0.2)}$$
$$h[n] = 5(0.5)^n u[n] - 4(0.2)^n u[n] = u[n]\left(5(2)^{-n} - 4(5)^{-n}u[n]\right)$$

h[n] = {1.00 1.70 1.09 0.59 0.31 0.16 }. And since the input $x[n]$ is a time shifted delta impulse,

$$y[n] = h[n] * x[n] = h[n-2]$$

And y[n] = {0.00 0.00 1.00 1.70 1.09 0.59 0.31 0.16 }.

B) (5p) Verify by solving y[n] directly using the difference equation or by using long division.

$$\begin{aligned}
y[n] &= x[n] + x[n-1] + 0.7y[n-1] - 0.1y[n-2] \\
y[0] &= 0+0+0-0=0 \\
y[1] &= 0+0+0-0=0 \\
y[1] &= 1+0+0-0=1 \\
y[2] &= 0+1+0.7-0=1.7 \\
y[3] &= 0+0+0.7\cdot 1.7-0.1\cdot 1 = 1.09 \\
y[3] &= 0+0+0.7\cdot 1.09-0.1\cdot 1.7 = 0.593
\end{aligned}$$

C) (10p) Consider 4 different signals, z-transforms of which have been represented on the z-plane below. Sketch approximate discrete time signals corresponding to those transforms. Explain the difference between discrete time frequency and continuous time frequency. In

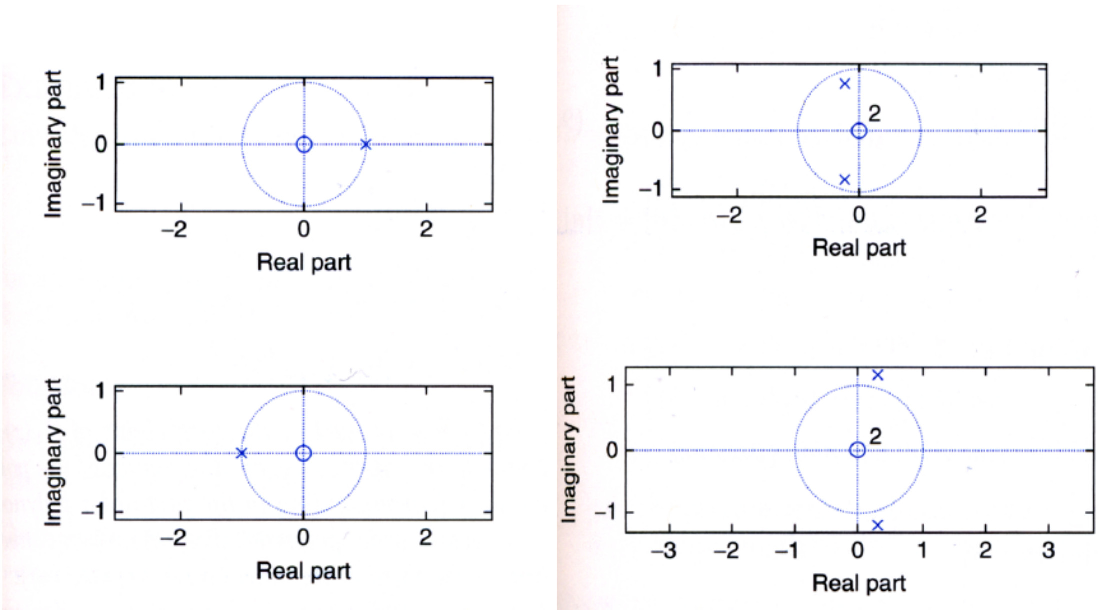

Figure 1: Q2c

principle a continuous time domain signal can have any frequency, but once this signal is discretized, the frequency content of the discrete signal is restricted by the sampling frequency. If Ω is the frequency of continuous time domain signal $x(t)$ and ω is the frequency of the discrete-time signal $x(nT_s)$, then

$$\omega = \Omega T_s$$

and for a discrete-time domain signal,sampled with a sampling time T_s (and corresponding sampling frequency $\Omega_s = 2\pi/T_s$), $\Omega \in [-\Omega_s/2 : \Omega_s/2]$; $\omega \in [-\pi : \pi]$

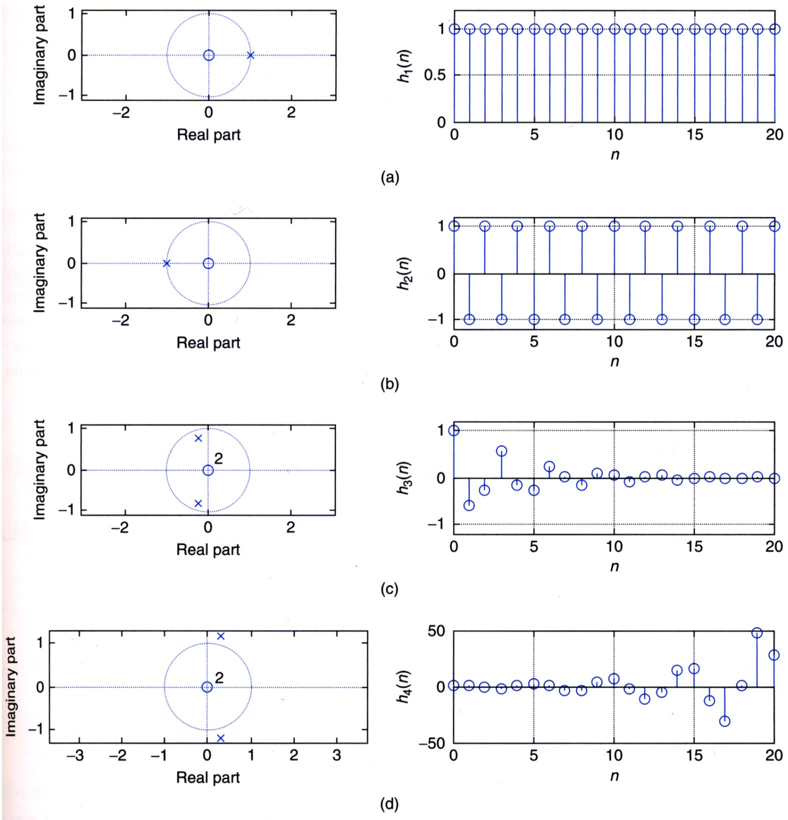

Figure 2: 1) zero frequency Amplitude = 1; 2) maximum frequency, the same amplitude; real exponentially decaying signal with frequency larger that the one in 4; exponentially increasing signal.

Q3 (25p) Train of delta impulses δ_T is defined by:

$$\delta_T = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

where T_s is the time delay between consecutive impulses.

A) **(10p)** Given that the Fourier transform $F_s(w)$ of sampled function $f_s(t)$ is given by :

$$\begin{aligned} f_s(t) &= f(t)\delta_T(t) \\ F_s(\omega) &= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s) \end{aligned}$$

explain how to reconstruct the continuous-time domain function $f(t)$ from $f_s(t)$. What criteria must be satisfied by $f_s(t)$ and/or $f(t)$ for this to be possible?

Full reconstruction is only possible for band limited signals ($F(\omega) = 0$ for $|\omega| < \omega_{max}$) which are sampled accordingly to Nenquist condition ($\omega_s > 2\omega_{max}$). In that case reconstruction can be done using a ideal low pass filter, with a cut-off frequency $\omega_s/2$ and the amplitude T_s. For such a filter, $F_s(\omega)H(\omega) = F(\omega)$. It can be shown that this corresponds to sinc interpolation, as such ideal filter will have a impulse response function given by sinc function. Filtered output will be convolution between sampled signal and "sinc" impulse response of the filter.

B) **(15p)** Define DTFT (Descrete Time Fourier Transform) and DFT (Discrete Fourier Transform) of the sampled signal $f_s(t)$ and calculate DTFT for

$$x[n] = \begin{cases} 1 & 0 < n < 3 \\ 0 & \text{otherwise} \end{cases}$$

Can you use the answer to write the expression for DFT of the same signal?.

DTFT can be calculated for discrete, aperiodic signal defined for all n. Resulting transform is periodic with respect to discrete frequency ω.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \sum_{n=1}^{2} x[n] e^{-j\omega n} = e^{-1j\omega} + e^{-2j\omega}$$

The answer can be used to calculate DFT. But in that case the signal $x[n]$ has to be finite in the time domain. We have to define a new signal, with periodicity N, where $N > 3$. For example:

$$\tilde{x}[n] = \begin{cases} 1 & 0 < n < 3 \\ 0 & 3 \le n < N \end{cases}$$

$$\tilde{X}[k] = \sum_{n=0}^{M-1} e^{-j2\pi kn/N} = e^{-j2\pi k/N} + e^{-j4\pi k/N}$$

This is defined for all k, but is of course periodic as well.

Fourier Series

$$f_p(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_0^T f_p(t) e^{-jn\omega_0 t} \mathrm{d}t$$

Fourier Transform

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \mathrm{d}\omega$$

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \mathrm{d}t$$

Fourier transform of a periodic function:

$$F(w) = \sum_{n=-\infty}^{\infty} \frac{2\pi}{T_0} F(n\omega_0)\delta(w - n\omega_0)$$

Q4 (25p) The scheme below illustrates a simple high pass filter.

A) **(10p)** Find the relationship between input $v_i(t)$ and the output $v_o(t)$ signals for this filter.

$$\begin{aligned}
v_i(t) &= iR + L\frac{\mathrm{d}i}{\mathrm{d}t} \\
V_i(s) &= I(s)R + sLI(s) \\
I(s) &= \frac{V_i(s)}{R + sL} \\
v_o(t) &= L\frac{\mathrm{d}i}{\mathrm{d}t} \\
V_o(s) &= sLI(s) \\
V_o(s) &= sL\frac{V_i(s)}{R + sL} \\
V_o(s)(R + sL) &= sLV_i(s) \\
Rv_o(t) + L\frac{\mathrm{d}v_0}{\mathrm{d}t} &= L\frac{\mathrm{d}v_i(t)}{\mathrm{d}t} \\
Rv_o(t) + L\frac{\mathrm{d}v_0}{\mathrm{d}t} &= L\frac{\mathrm{d}v_i(t)}{\mathrm{d}t}
\end{aligned}$$

B) **(10p)** You would now like to design discrete time-domain (DSP) filter with similar characteristics. Derive the difference equation for this system and draw a block diagram for DSP filter.

$$\begin{aligned}
x[n] = v_i(nt_s) \quad & y[n] = v_o(nt_s) \\
Ry[n] + L\frac{y[n] - y[n-1]}{t_s} &= L\frac{x[n] - x[n-1]}{t_s} \\
y[n](R + L/t_s) - y[n-1]\frac{L}{t_s} &= \frac{L}{t_s}(x[n] - x[n-1]) \\
y[n]\left(\frac{Rt_s + L}{t_s}\right) &= y[n-1]\frac{L}{t_s} + \frac{L}{t_s}(x[n] - x[n-1]) \\
y[n] &= (x[n] - x[n-1] + y[n-1])\frac{L}{Rt_s + L} \\
y[n] &= (x[n] - x[n-1] + y[n-1])\frac{1}{\alpha} \\
\alpha &= \frac{Rt_s + L}{L}
\end{aligned}$$

C) **(5p)** Write the z-transform of the transfer function. Calculate impulse response $h[n]$ with $0 \leq n < 3$ for this DSP filter using a method of choice.

$$\begin{aligned}
\alpha y[n] - y[n-1] &= x[n] - x[n-1] \\
\alpha Y(z) - Y(z)z^{-1} &= X(z) - X(z)z^{-1} \\
Y(z)\left(\alpha - z^{-1}\right) &= X(z)\left(1 - z^{-1}\right) \\
Y(z)/X(z) = H(z) &= \frac{1 - z^{-1}}{\alpha - z^{-1}} \\
H(z) &= \frac{z - 1}{\alpha z - 1}
\end{aligned}$$

$$
\begin{aligned}
x[n] &= \delta[n] \\
y[0] &= \frac{1}{\alpha} \\
y[1] &= \frac{1}{\alpha^2} - \frac{1}{\alpha} \\
y[2] &= \left(\frac{1}{\alpha^2} - \frac{1}{\alpha}\right)\frac{1}{\alpha}
\end{aligned}
$$

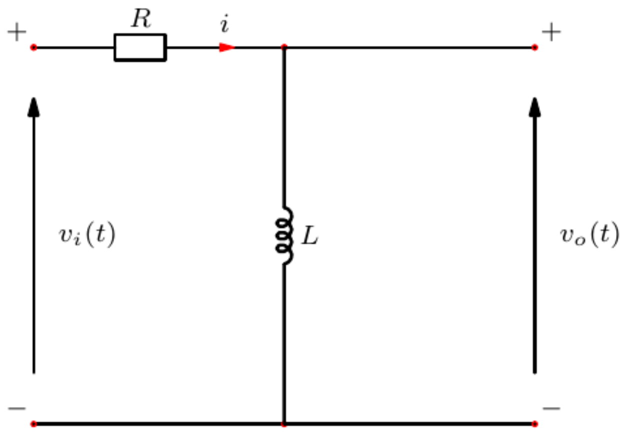

Figure 3: Q4

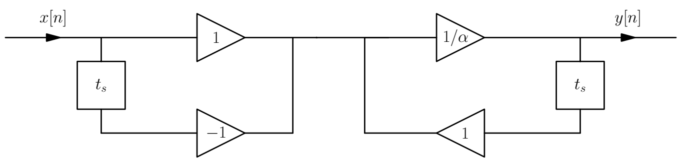

Figure 4: Answer Q4. Diagram

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KONT EXAM (English)
TFY4280 Signal Processing

August 2013

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Q1 (30p)

A) The input of an LTI system is

$$x(t) = \epsilon(t) - 2\epsilon(t-1) + \epsilon(t-2)$$

where $\epsilon(t)$ is the unit step function. If the Laplace transform of the output is given by

$$Y(s) = \frac{(s+2)(1-e^{-s})^2}{s^2(s+1)^2}$$

Determine the transfer function of the system.
The transfer function $H(s) = Y(s)/X(s)$ is found by taking the Laplace transform of the input signal.

$$\begin{aligned}
X(s) &= \frac{1}{s} - 2e^{-s}\frac{1}{s} + e^{-2s}\frac{1}{s} = \frac{1}{s}\left(1 - 2e^{-s} + e^{-2s}\right) = \frac{1}{s}\left(1 - e^{-s}\right)^2 \\
Y(s) &= \frac{(s+2)(1-e^{-s})^2}{s^2(s+1)^2} \\
H(s) &= \frac{Y(s)}{X(s)} = \frac{(s+2)(1-e^{-s})^2}{s^2(s+1)^2}\frac{s}{(1-e^{-s})^2} = \frac{s+2}{s(s+1)^2}
\end{aligned}$$

We can then find $h(t)$ by finding the inverse transform of $H(s)$

$$\begin{aligned} H(s) &= \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{s} = -\frac{1}{(s+1)^2} - \frac{2}{(s+1)} + \frac{2}{s} \\ h(t) &= 2\epsilon(t) - 2e^{-t}\epsilon(t) - te^{-t}\epsilon(t) \end{aligned}$$

B) Find the the unit step response $s(t)$ for a system described by:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$$

if $y(0) = 1$; $\dot{y}(0) = 0$

The Laplace transform of the differential equation gives:

$$\begin{aligned} \left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = X(s) \\ Y(s)\left(s^2 + 3s + 2\right) - (s+3) = X(s) \\ Y(s) = X(s)\frac{1}{(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)} \end{aligned}$$

and since $X(s) = \mathscr{L}\{\epsilon(t)\} = \frac{1}{s}$, we have

$$\begin{aligned} Y(s) &= \frac{1}{s(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)} \\ Y(s) &= \frac{1+s^2+3s}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \\ Y(s) &= \frac{0.5}{s} + \frac{1}{s+1} - \frac{0.5}{s+2} \\ y(t) &= 0.5\epsilon(t) + e^{-t}\epsilon(t) - 0.5e^{-2t}\epsilon(t)\,. \end{aligned}$$

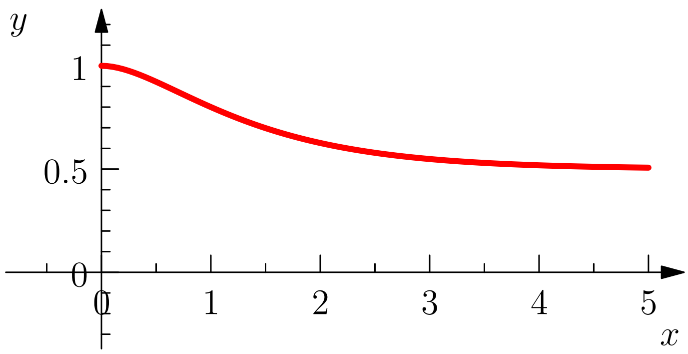


Figure 5: $s(t)$ for initial condition: $y(0) = 1$ and $\dot{y}(0) = 0$.

Q2 (30p)

A) Two signals x[n] and y[n] are given by:

$$\begin{aligned} x[n] &= \begin{cases} 5-n & 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases} \\ y[n] &= \begin{cases} 1 & 2 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

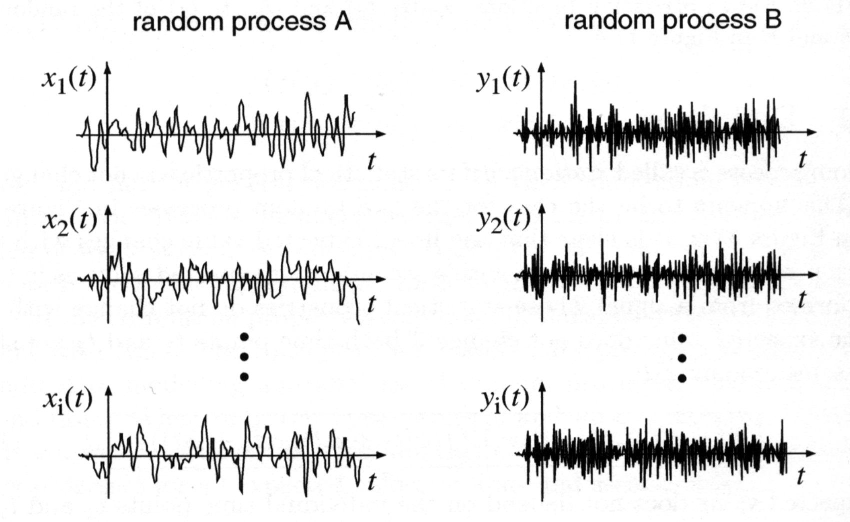
1. Sketch x[n] and y[n]
2. Sketch x[n - k] for k = 3 and k = -3
3. Sketch x[-n]
4. Sketch x[n]y[n]

B) Convolve signals $x[n] = [2\ 5\ \underline{3}\ \text{-}1\ 0\ 1]$ and $y[n] = [\text{-}2\ \underline{1}\ \text{-}3\ 4\ 2]$ with indices ranging between -2:3 and -1:3, respectively.

Q3 (20p)

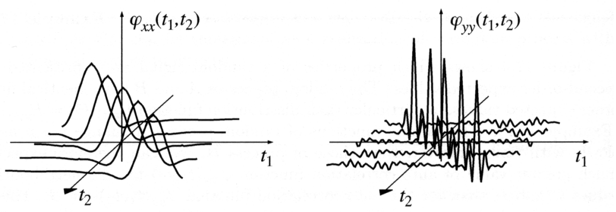
A) What is an "ergodic random process"; what is a "stationary random process" Ergodic:Expected values (ensemble average) can be replaced by time average of one sample function. Stationary: second order expected values only depends on the difference in the observation time points $\tau = t_2 - t_1$ and not on a particular choice of t_1 and $t_2 = t_1 + \tau$

B) Define auto-correlation function (ACF). Sketch ACF for two signals, for which few sample functions are shown below:



ACF:

$$\varphi_{xx}(t_1, t_2) = E\{x(t_1)x(t_2)\}$$



Q4 (20p) Given the following unilateral z-transforms:

i)

$$X_1(z) = \frac{0.5z^2}{(z-1)(z-0.5)}$$

ii)

$$X_2(z) = \frac{0.5z}{(z-1)(z-0.5)}$$

- Find the inverse z-transforms.
- Evaluate a few values of $x_1[n]$ and $x_2[n]$.

$$\frac{0.5z^2}{(z-1)(z-0.5)} = \left[\frac{k_1 z}{z-1} + \frac{k_2 z}{z-0.5}\right]$$

To get k1, we multiply both sizes by $(z-1)$ and set $z=1$

$$\frac{0.5z^2}{(z-0.5)} = [k_1 z]|_{z=1}$$
$$k_1 = 1$$

To get k2, we multiply both sizes by $(z-0.5)$ and set $z=0.5$

$$\frac{0.5z^2}{(z-1)} = [k_2 z]$$
$$k_2 = -0.5$$

$$\left[\frac{z}{z-1} - \frac{0.5z}{z-.5}\right] = \frac{z^2-0.5z-0.5z^2+0.5z}{(z-1)(z-0.5)} = 0.5\frac{z^2}{(z-1)(z-0.5)}_{OK}$$

So,

$$x[n] = u[n] - 0.5(0.5)^n u[n] = u[n] - (0.5)^{n+1}u[n]$$

For ii)

$$\frac{0.5z}{(z-1)(z-0.5)} = \left[\frac{k_1 z}{z-1} + \frac{k_2 z}{z-0.5}\right]$$

To get k1, we multiply both sizes by $(z-1)$ and set $z=1$

$$\frac{0.5z}{(z-0.5)} = [k_1 z]|_{z=1}$$
$$k_1 = 1$$

To get k2, we multiply both sizes by $(z-0.5)$ and set $z=0.5$

$$\frac{0.5z}{(z-1)} = [k_2 z]|_{z=0.5}$$
$$k_2 = -1$$

$$\left[\frac{z}{z-1} - \frac{z}{z-.5}\right] = 0.5\frac{z^2-0.5z-z^2+z}{(z-1)(z-0.5)} = 0.5\frac{z}{(z-1)(z-0.5)}_{OK}$$

$$x[n] = u[n] - (0.5)^n u[n]$$

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} < \mathrm{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
$te^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$t^ne^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
$\cos(\omega_0 t)\varepsilon(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$e^{-at}\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$t\cos(\omega_0 t)\varepsilon(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\} > 0$
$t\sin(\omega_0 t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\} > 0$

Appendix B.2 Properties of the Bilateral Laplace Transform

	$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(s) + BX_2(s)$	ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$
Delay	$x(t-\tau)$	$e^{-s\tau}X(s)$	not affected
Modulation	$e^{at}x(t)$	$X(s-a)$	Re$\{a\}$ shifted by Re$\{a\}$ to the right
'Multiplication by t', Differentiation in the frequency domain	$tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$sX(s)$	ROC $\supseteq$ ROC$\{X\}$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	ROC $\supseteq$ ROC$\{X\}$ $\cap\{s : \mathrm{Re}\{s\} > 0\}$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot{\delta}(t)$	$j\omega$
$\frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$	$\text{Ш}\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\mathrm{rect}(at)$	$\frac{1}{\lvert a\rvert}\mathrm{si}\left(\frac{\omega}{2a}\right)$
$\mathrm{si}(at)$	$\frac{\pi}{\lvert a\rvert}\mathrm{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\mathrm{sign}(\omega)$
$\mathrm{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
$e^{-\alpha\lvert t\rvert},\ \alpha > 0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-a^2t^2}$	$\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a \in \mathbb{R}\backslash\{0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega) \cdot X_2(j\omega)$
Multiplication	$x_1(t) \cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$

Appendix B.6 Properties of the z-Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k]+bx_2[k]$	$aX_1(z)+bX_2(z)$	ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$
Delay	$x[k-\kappa]$	$z^{-\kappa}X(z)$	ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	ROC$=\left\{z \,\middle\vert\, \frac{z}{a} \in \text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z\frac{dX(z)}{dz}$	ROC$\{x\}$; separate consideration of $z=0$
Time inversion	$x[-k]$	$X(z^{-1})$	ROC$=\{z\,\vert z^{-1}\in$ROC$\{x\}\}$
Convolution	$x_1[k]*x_2[k]$	$X_1(z)\cdot X_2(z)$	ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$
Multiplication	$x_1[k]\cdot x_2[k]$	$\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z-Transform Pairs

$x[k]$	$X(z)=\mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z\in\mathbb{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$\vert z\vert>1$
$a^k\varepsilon[k]$	$\frac{z}{z-a}$	$\vert z\vert>\vert a\vert$
$-a^k\varepsilon[-k-1]$	$\frac{z}{z-a}$	$\vert z\vert<\vert a\vert$
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$\vert z\vert>1$
$ka^k\varepsilon[k]$	$\frac{az}{(z-a)^2}$	$\vert z\vert>\vert a\vert$
$\sin(\Omega_0 k)\varepsilon[k]$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\vert z\vert>1$
$\cos(\Omega_0 k)\varepsilon[k]$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\vert z\vert>1$