

Q1 DTFT and DFT (25p)

1. Explain what is described by the term “discrete frequency”.

Answer: “discrete frequency” ω is a frequency space used to describe the frequency spectrum of a discrete signal. ω can be both continuous (as for aperiodic discrete signals and DTFT) or discrete (as for periodic discrete signals and DFT). Discrete frequency is connected to real frequency by $\Omega = \omega t_s$, where t_s is the sampling time of the signal in the time domain.

2. Find DTFT of the following discrete time signal:

$$x_1[n] = a^n u[n]$$

where $u[n]$ is the discrete unit step function. Use $t_s = 1\text{ms}$.

Answer:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} = \sum_{n=0}^{\infty} a^n e^{-jn\omega} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

We can for example, use the Euler formula to find the amplitude and the phase:

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}} = \frac{1}{1 - a \cos(\omega) + aj \sin(\omega)} = \\ &= \frac{1 - a \cos(\omega) - aj \sin(\omega)}{(1 - a \cos(\omega))^2 - (ja \sin(\omega))^2} = \\ &= \frac{1 - a \cos(\omega) - aj \sin(\omega)}{1 - 2a \cos(\omega) + a^2 \cos^2(\omega) + a^2 \sin^2(\omega)} = \\ &= \frac{1 - a \cos(\omega) - aj \sin(\omega)}{1 - 2a \cos(\omega) + a^2} \\ |X(e^{j\omega})| &= \frac{1}{\sqrt{(1 - 2a \cos(\omega) + a^2)}} \end{aligned}$$

3. Sketch the amplitude response for $a = 0.8$ using discrete frequency on the x-axis.

Answer: See Figure 1.

4. Sketch the amplitude response for $a = 0.8$ using frequency on the x-axis.

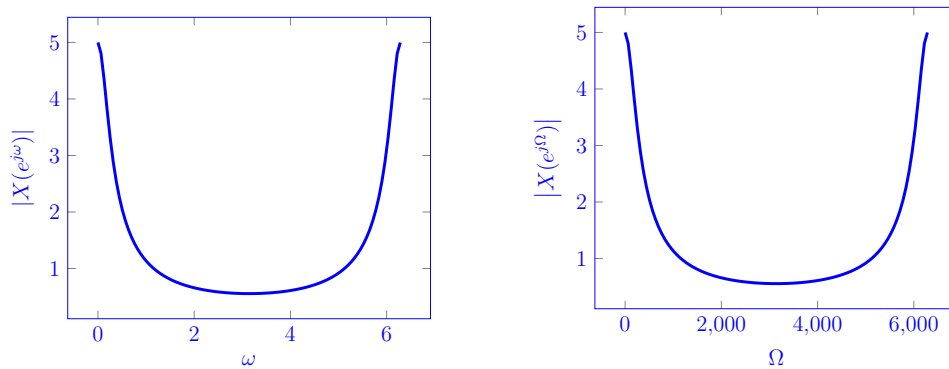


Figure 1: $X(e^{j\omega})$ plotted for discrete frequency ω between 0 and 2π and frequency Ω between 0 and Ω_s .

Answer:

$$\Omega = \omega t_s$$

so the frequency range should be $\Omega \in \left(0, \frac{2\pi}{t_s}\right)$, $\Omega \in (0, \Omega_s)$ or $\Omega \in (-\Omega_s/2, \Omega_s/2)$

5. What would you need to be able to calculate DFT of the same signal? What would be the main difference between DFT and DTFT? Please explain.

Answer: DFT is defined for signals which are discrete and periodic in the time domain. We therefore would need to define a periodic signal

$$\begin{aligned} x_2[n] &= a^n u[n] \quad 0 \leq n < N \\ x_2[n+N] &= x_2[n] \end{aligned}$$

For such signal, DFT is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \quad 0 \leq k \leq N-1 \quad (1)$$

and the discrete frequency is now discrete and given by $\omega_k = 2\pi k/N$. DFT is discrete in the frequency domain, defined only for integer values of k and periodic with periodicity of 2π .

Q2 System output and frequency response. (25p)

1. A LTI system is characterized by the impulse response function given below. Calculate and plot the response of that system to a unit step input $x_1(t) = \varepsilon(t)$ and to a delta impulse input $x_2(t) = \delta(t)$.

$$h(t) = \varepsilon(t)e^{-t} - 2e^{-3t}\varepsilon(t)$$

Also here, $\varepsilon(t)$ is the unit step function.

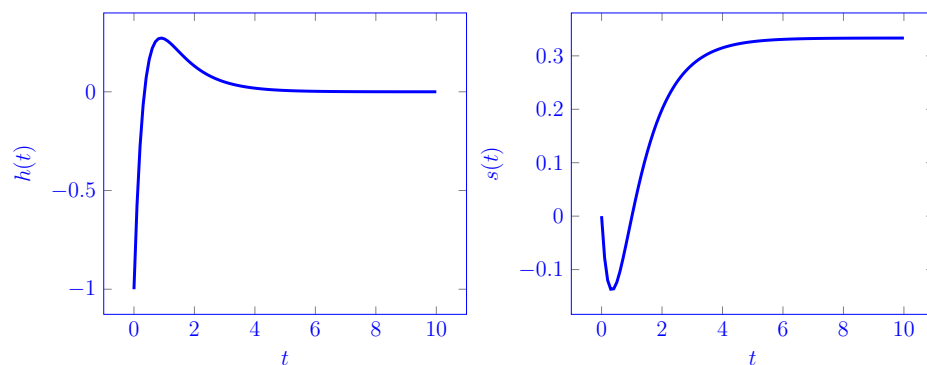
Answer:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+1} - \frac{2}{s+3} = \frac{s+3-2s-2}{(s+3)(s+1)} = \frac{1-s}{(s+3)(s+1)}$$

$$Y(s) = X(s)H(s)$$

$$Y(s) = \frac{1-s}{(s+3)(s+1)} \frac{1}{s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} = \frac{1}{3s} - \frac{1}{s+1} + \frac{2}{3(s+3)}$$

$$y(t) = \varepsilon(t) \left[\frac{1}{3} - e^{-t} + \frac{2}{3}e^{-3t} \right]$$



2. Calculate frequency response of this system for $\omega = 0$ and $\omega = 2\pi s^{-1}$. Explain how you would calculate the frequency response for any given frequency.

Answer: Frequency response is given by

$$H(j\omega) = H(s)|_{s=j\omega}$$

Where $H(s)$ has been calculated above. To get an general expression for the frequency response, we need to substitute $s = j\omega$ and calculate $H(j\omega)$. For the two frequencies given in the problem text, one can use the simple expression for $H(s)$ obtained above.

$$H(j\omega) = \frac{1}{1+j\omega} - \frac{2}{3+j\omega}$$

For $\omega = 0$

$$|H(j\omega)| = 1 - \frac{2}{3} = \frac{1}{3}$$

For $\omega = 2\pi$

$$\begin{aligned} H(2\pi j) &= \frac{1}{1 + 2\pi j} - \frac{2}{3 + 2\pi j} = \frac{1 - 2\pi j}{1 + 4\pi^2} - \frac{6 + 4\pi j}{9 + 4\pi^2} = \\ &= 0.0247 - 0.155j - 0.1238 + 0.2592j = \\ &= -0.0991 + 0.1040j \end{aligned}$$

$$|H(2\pi j)| = 0.1436$$

Q3 Stochastic Signals (25p)

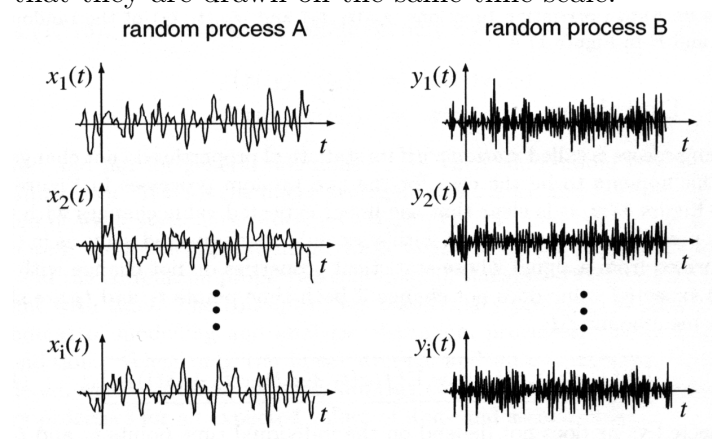
1. What is an “ergodic random process”? What is a “stationary random process”?

Answer: A stationary random process for which the time-averages of each sample function are the same as the ensemble averages is called ergodic random process.

A random process is stationary if its **2nd order expected values** only depend on the difference τ between t_1 and t_2 , that is:

$$E \{f(x(t_1), x(t_2))\} = E \{f(x(t_1), x(t_1 + \tau))\} \quad (2)$$

2. Define auto-correlation function (ACF). Sketch ACF for two signals, for which few sample functions are shown below, assuming that they are drawn on the same time scale.



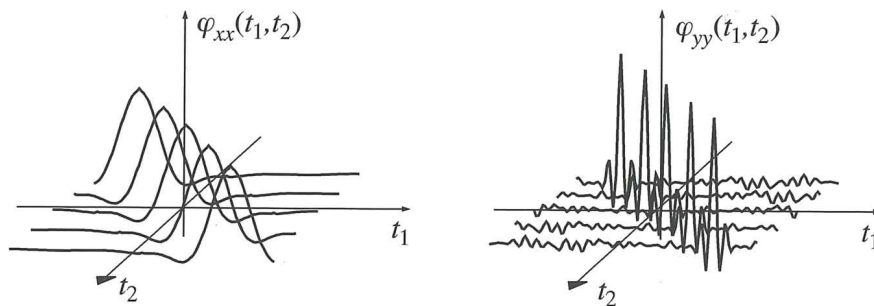
Answer:

$$\varphi_{xx}(t_1, t_2) = E \{x(t_1)x(t_2)\} \quad (3)$$

For a stationary process, this simplifies to

$$\varphi_{xx}(t_1, t_2) = \varphi_{xx}(\tau) = E \{x(t_1)x(t_1 + \tau)\} \quad (4)$$

In general ACF is a 2D function defined for any combination of t_1 and t_2 . For stationary process, it becomes a 1D function of τ . Slower rate of change for the process A, would result in slower decay of the ACF, as illustrated on the figure below.



Q4 Filters. (25p)

1. For time-discrete systems, filters are often characterized as IIR or FIR. Explain what is described by these terms.

Answer:

IIR: infinite impulse response, system described by a recursive equation containing both $x[k]$ and $y[k]$ terms (input and output). $H(z)$ has both zeros and poles. $h[k]$ is given by a recursive equation.

$$\begin{aligned} \sum_{n=0}^N a_n y[k-n] &= \sum_{n=0}^M b_n x[k-n] \\ y[k] &= \sum_{n=0}^M b_n x[k-n] - \sum_{n=1}^N a_n y[k-n] \end{aligned}$$

If we put $x[k] = \delta[k]$, then we can see that

$$h[k] = \sum_{n=0}^M b_n \delta[k-n] - \sum_{n=1}^N a_n h[k-n]$$

So, the impulse response function at time point k depends on both b_k coefficient (delta function in the first sum is not zero only for $k = n$), but also on previous values of the impulse response function ($h[k-1]$, $h[k-2]$, etc)

FIR: finite impulse response.

$$y[k] = \sum_{n=0}^M b_n x[k-n]$$

If we put $x[k] = \delta[k]$, then we can see that

$$h[k] = \sum_{n=0}^M b_n x[k-n] = \{b_0, b_1, b_2, \dots, b_M\}$$

2. For which of these filter types, can we use discrete convolution to calculate output for an input signal similar to $x[n] = \{1, 1, 1, 1, 1, 1\}$? Do not calculate but explain.

Answer: Only for FIR, as in as to do an exact calculation we would need to include infinite number of terms for $h[n]$ in the case of IIR filter.

3. A low pass filter is described by the difference equation given below. Use zeros/poles diagram on an appropriate frequency plane to illustrate that this system is indeed a low pass filter.

$$y[n] = \alpha x[n] + (1 - \alpha)y[n-1]$$

where α is a constant and $0 < \alpha < 1$

Answer: To find zeros/poles of the transfer function we need to take z-transform of the difference equation and calculate $H(z)$.

$$H(z) = \frac{\alpha z}{z - (1 - \alpha)}$$

This transform has a pole at $z = 1 - \alpha$ which is located on the horizontal axis, on the right hand side of the z-plane and for α approaching zero, the pole will be located close to the DC frequency, resulting in a large amplitude of the transfer function for these frequencies.

4. Is this a IIR or FIR filter? Please explain.

Answer: It is an IIR filter. One can see it for example by calculating few terms of the impulse response function.

$$\begin{aligned}h[0] &= \alpha \\h[1] &= \alpha(1 - \alpha) \\h[2] &= \alpha(1 - \alpha)^2 \\h[2] &= \alpha(1 - \alpha)^3 \\&\dots\end{aligned}$$

One could transfer this filter to FIR by setting

$$h[n] = 0 \quad \text{for } n > N$$

Where $N > 0$.

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\text{Re}\{s\} > 0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} < \text{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\text{Re}\{s\} > 0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}\{s\} > 0$
$t e^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$t^n e^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\text{Re}\{s\} > 0$
$\cos(\omega_0 t)\varepsilon(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\text{Re}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$e^{-at}\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$t\cos(\omega_0 t)\varepsilon(t)$	$\frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2}$	$\text{Re}\{s\} > 0$
$t\sin(\omega_0 t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2 + \omega_0^2)^2}$	$\text{Re}\{s\} > 0$

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\delta(t)$	$j\omega$
$\frac{1}{T} \text{III}\left(\frac{t}{T}\right)$	$\text{III}\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\text{rect}(at)$	$\frac{1}{ a } \text{si}\left(\frac{\omega}{2a}\right)$
$\text{si}(at)$	$\frac{\pi}{ a } \text{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi \text{sign}(\omega)$
$\text{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
$e^{-\alpha t }, \alpha > 0$	$\frac{2\alpha}{\alpha^2 + \omega^2}$
$e^{-a^2 t^2}$	$\frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}}$

Appendix B.2 Properties of the Bilateral Laplace Transform

$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
Linearity $Ax_1(t) + Bx_2(t)$	$AX_1(s) + BX_2(s)$	$\text{ROC} \supseteq \text{ROC}\{X_1\} \cap \text{ROC}\{X_2\}$
Delay $x(t - \tau)$	$e^{-s\tau}X(s)$	not affected
Modulation $e^{at}x(t)$	$X(s - a)$	$\text{Re}\{a\}$ shifted by $\text{Re}\{a\}$ to the right
'Multiplication by t ', Differentiation in the frequency domain $tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain $\frac{d}{dt}x(t)$	$sX(s)$	$\text{ROC} \supseteq \text{ROC}\{X\}$
Integration $\int_{-\infty}^t x(\tau) d\tau$	$\frac{1}{s}X(s)$	$\text{ROC} \supseteq \text{ROC}\{X\} \cap \{s : \text{Re}\{s\} > 0\}$
Scaling $x(at)$	$\frac{1}{ a }X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$
Delay	$x(t - \tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
'Multiplication by t ', Differentiation in the frequency domain $tx(t)$	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain $\frac{dx(t)}{dt}$	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^t x(\tau) d\tau$	$X(j\omega) \left[\pi\delta(\omega) + \frac{1}{j\omega} \right]$ $= \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{j\omega}{a}\right), \quad a \in \mathbb{R} \setminus \{0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega) \cdot X_2(j\omega)$
Multiplication	$x_1(t) \cdot x_2(t)$	$\frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty} x(t) ^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) ^2 d\omega$

Appendix B.6 Properties of the z -Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k] + bx_2[k]$	$aX_1(z) + bX_2(z)$	$\text{ROC} \supseteq \text{ROC}\{X_1\} \cap \text{ROC}\{X_2\}$
Delay	$x[k - \kappa]$	$z^{-\kappa}X(z)$	$\text{ROC}\{x\}$; separate consideration of $z = 0$ and $z \rightarrow \infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	$\text{ROC} = \left\{z \mid \frac{z}{a} \in \text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z \frac{dX(z)}{dz}$	$\text{ROC}\{x\}$; separate consideration of $z = 0$
Time inversion	$x[-k]$	$X(z^{-1})$	$\text{ROC} = \{z \mid z^{-1} \in \text{ROC}\{x\}\}$
Convolution	$x_1[k] * x_2[k]$	$X_1(z) \cdot X_2(z)$	$\text{ROC} \supseteq \text{ROC}\{x_1\} \cap \text{ROC}\{x_2\}$
Multiplication	$x_1[k] \cdot x_2[k]$	$\frac{1}{2\pi j} \oint X_1(\zeta) X_2\left(\frac{z}{\zeta}\right) \frac{1}{\zeta} d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z -Transform Pairs

$x[k]$	$X(z) = \mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z \in \mathbf{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$ z > 1$
$a^k \varepsilon[k]$	$\frac{z}{z-a}$	$ z > a $
$-a^k \varepsilon[-k-1]$	$\frac{z}{z-a}$	$ z < a $
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$ z > 1$
$ka^k \varepsilon[k]$	$\frac{az}{(z-a)^2}$	$ z > a $
$\sin(\Omega_0 k) \varepsilon[k]$	$\frac{z \sin \Omega_0}{z^2 - 2z \cos \Omega_0 + 1}$	$ z > 1$
$\cos(\Omega_0 k) \varepsilon[k]$	$\frac{z(z - \cos \Omega_0)}{z^2 - 2z \cos \Omega_0 + 1}$	$ z > 1$