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Solution set for final exam in TFY 4305

December 12, 2009

Problem 1

$$a) \quad \dot{x} = 0 = y(x+1) \Rightarrow x = -1, y = 0$$

$$\dot{y} = 0 = x(1+y^3) \Rightarrow x = 0, y = -1$$

$$\Rightarrow \underline{(x^*, y^*) = (0, 0) \text{ or } (-1, -1)}$$

b)

$$\text{Jacobian matrix: } J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x+1 \\ 1+y^3 & 3xy^2 \end{pmatrix}$$

at (0,0)

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \left. \begin{array}{l} \tau = 0 = \lambda_1 + \lambda_2 \\ \Delta = -1 = \lambda_1 \cdot \lambda_2 \end{array} \right\}$$

$$\Rightarrow \lambda = \pm 1 \Rightarrow \underline{\text{saddle point}}$$

At $(-1, -1)$,

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$$J = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\tau = \lambda_1 + \lambda_2 = -4$$

$$\Delta = \lambda_1 \cdot \lambda_2 = 3$$

$$\Rightarrow \lambda_1 = -1, \lambda_2 = -3$$

Stable node.

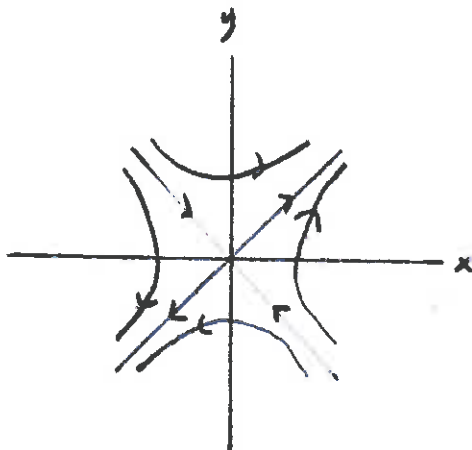
c) Eigen directions for $(0,0)$

$$J \vec{v} = \lambda \vec{v}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Rightarrow \lambda = 1; \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1; \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



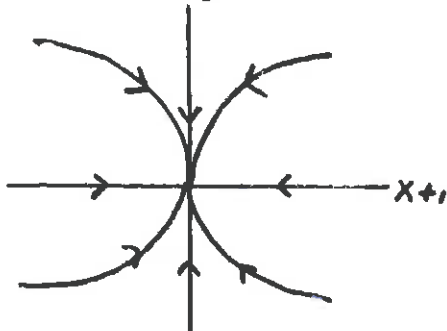
Eigen directions for $(-1, -1)$

$$J \vec{v} = \lambda \vec{v}$$

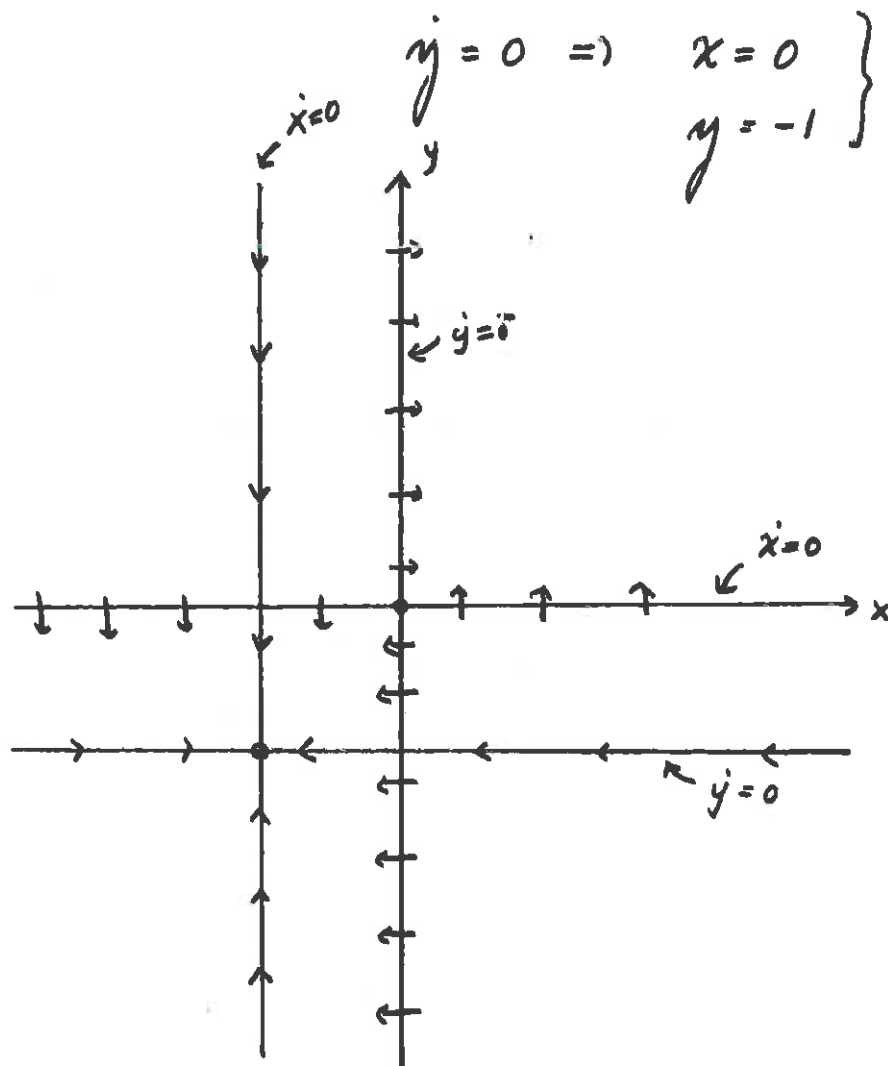
$$\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Rightarrow \lambda = -1; \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = -3; \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

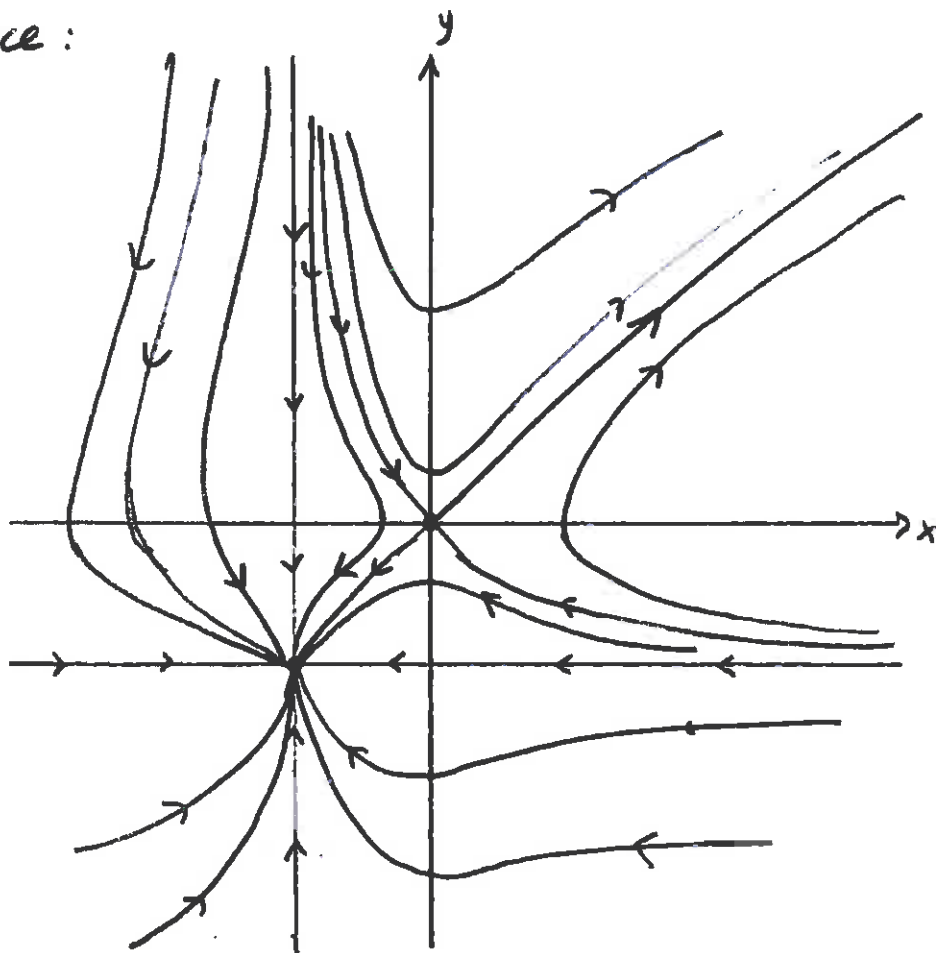


Nullclines : $\dot{x} = 0 \Rightarrow \left. \begin{array}{l} y = 0 \\ x = -1 \end{array} \right\}$



The signs have been placed using the flows near the two fixed points.

We now combine all the information⁴
into a complete sketch of phase
space:



Problem 2

$$\begin{aligned} \text{a)} \quad & x = -2y^2 \\ & xy = x^2y \end{aligned}$$

} We see immediately
that $(x,y) = (0,0)$
is a solution.

Is it the only one?

Suppose $x \neq 0$. Then $x = -2y^2 \Rightarrow y \neq 0$

$$\left. \begin{aligned} xy = x^2 y &\Rightarrow \\ x = 1 & \\ y^2 = -\frac{1}{2} & \end{aligned} \right\}$$

$\Rightarrow$ No solution.

Suppose $y \neq 0$. Then $-2y^2 = x \Rightarrow x \neq 0$;

And there is no solution.

$\Rightarrow$ $(0,0)$ is the only fixed point.

b) I have a Liapunov function, it must satisfy

(1) $V(x^*, y^*) = 0$

(2) $V(x, y) > 0 \quad (x, y) \neq (x^*, y^*)$

(3) $\dot{V}(x, y) < 0 \quad (x, y) \neq (x^*, y^*)$.

Using the proposed function, (1) and (2) are immediately fulfilled.

We now investigate (3):

$$\begin{aligned}\frac{dV}{dt} &= \frac{d}{dt} \{x^2 + ay^2\} = 2x\dot{x} + 2ay\dot{y} \\ &= 2x\{-x - 2y^2\} + 2ay\{xy - x^2y\} \\ &= -2x^2 - 4xy^2 + 2axy^2 - 2ax^2y^2\end{aligned}$$

If we now set $a=2$, we get

$$\frac{dV}{dt} = -2x^2 - 4x^2y^2 < 0 \quad \text{when } (x, y) \neq (0, 0).$$

Hence, $V(x, y) = x^2 + 2y^2$ is a Liapunov function and the fixed point is stable.

Problem 3

a)

$$\frac{\partial H}{\partial \phi_n} = \frac{\partial}{\partial \phi_n} \left\{ \frac{1}{2} (\phi_{n+1} - \phi_n)^2 + \frac{1}{2} (\phi_n - \phi_{n-1})^2 + \frac{a}{2} (\phi_n^2 - 1)^2 \right\}$$

↑
These are the only terms in H containing ϕ_n .

$$\frac{\partial H}{\partial \phi_n} = -(\phi_{n+1} - \phi_n) + (\phi_n - \phi_{n-1}) + \frac{a}{2} (\phi_n^2 - 1) 2\phi_n = 0$$

⇒

$$\underline{\phi_{n+1} + \phi_{n-1} - 2\phi_n = a\phi_n(\phi_n^2 - 1)}$$

b)

Set $x_n = \phi_n$ and $y_n = \phi_{n-1}$

Then, the equation here may be written

$$x_{n+1} + y_n - 2x_n = a x_n (x_n^2 - 1)$$

$$\Rightarrow$$

$$\underline{x_{n+1} = 2x_n + a x_n (x_n^2 - 1) - y_n}$$

From $y_n = \phi_{n-1}$, we have that

$$\underline{y_{n+1} = \phi_n = x_n}$$

Hence,

$$\underline{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2x_n + a x_n (x_n^2 - 1) - y_n \\ x_n \end{pmatrix}}$$

$$= T \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

c) We consider the Jacobian of T :

$$J = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} \end{pmatrix} = \begin{pmatrix} 2-a+3ax_n^2 & -1 \\ 1 & 0 \end{pmatrix}$$

The determinant is:

$$\Delta = 1 \Rightarrow \underline{\text{Area preserving.}}$$

d) The fixed point equation is:

$$\left. \begin{aligned} x^* &= 2x^* + ax^*(x^{*2}-1) - y^* \\ y^* &= x^* \end{aligned} \right\} \Rightarrow$$

$$ax^{*3} - ax^* \Rightarrow x^* = 0, \pm 1$$

$$\Rightarrow y^* = 0, \pm 1$$

$$\underline{(x^*, y^*) = (0, 0), (1, 1), (-1, -1)}$$

e) (0,0):

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$$\begin{vmatrix} 2-a-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(2-a-\lambda) + 1 = 0$$

$$\underline{\lambda^2 - (2-a)\lambda + 1 = 0}$$

$$\Rightarrow \underline{\lambda = \frac{1}{2} \{ 2-a \pm \sqrt{a^2 - 4a} \}}$$

When $a < 4$, they are complex numbers on the unit circle.

(1,1) and (-1,-1):

$$\begin{vmatrix} 2+2a-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(2+2a-\lambda) + 1 = 0$$

$$\underline{\lambda^2 - 2(1+a)\lambda + 1 = 0}$$

$$\Rightarrow \underline{\lambda_{\pm} = 1+a \pm \sqrt{2a+a^2}}$$

$\lambda_+ > 1$: Unstable eigendirection for all a .

$\lambda_- < 1$: stable eigendirection for all a .

f) When $a \rightarrow 4$, the two complex eigenvalues collide. $\lambda = -1$ at this point. We are therefore dealing with a period doubling at this point.
