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Continuation Exam TFY4345: Classical Mechanics
August 13th 2012
09.00-13.00
English

The exam consists of 4 problems. Each problem counts for in total 25% of the total weight of the exam, but each sub-exercise **(a)**, **(b)**, etc. does not necessarily count equally.

Read each problem carefully in order to avoid unnecessary mistakes.

Allowed material to use at exam: C.

- Approved, simple calculator.
- K. Rottmann: Matematisk formelsamling.
- K. Rottmann: Mathematische Formelsammlung. Barnett & Cronin: Mathematical Formulae.

Also consider the Supplementary Material on the last page of this exam.

PROBLEM 1

The brachistochrone problem consists of finding the curve between two points such that the time required for a particle to move between them becomes minimal.

(a) Solve the brachistochrone problem where the coordinate axes are laid as in Fig. 1. The particle starts from the origin, at rest, when $t = 0$. Find a closed analytical form for the coordinates x and y.

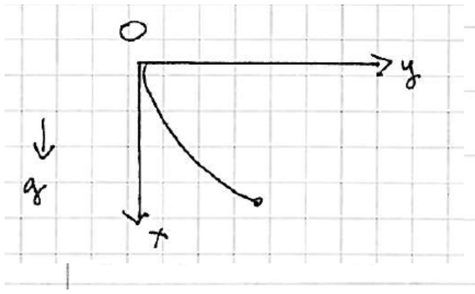

FIG. 1: (Color online). The system under consideration in a).

(b) Assume that the initial velocity is now $\mathbf{v_0}$, making an angle $\pi/4$ with the y-axis at $t = 0$. Show that the brachistochrone curve is determined from the equation

$$[y'(x)]^2 = f(v_0, g, x) \tag{1}$$

and identify the function $f(v_0, g, x)$ where $v_0 = |\mathbf{v_0}|$.

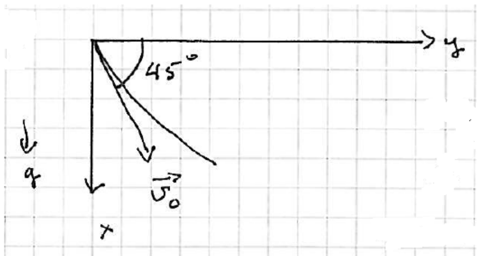

FIG. 2: (Color online). The system under consideration in b).

Consider now a different problem.

(c) Consider a rigid body rotating around a fixed point of the body. Find an expression for the time variation of the kinetic energy T as a function of the angular velocity vector ω and the total torque vector τ acting on the body. In effect, find dT/dt as a function of ω and τ.

PROBLEM 2

(a) A meson of mass m_π at rest disintegrates into a meson of mass m_μ and a neutrino of zero mass. Derive an expression for the kinetic energy of the μ-meson expressed only with the given masses and the speed of light c.

(b) Explain in detail the meaning of the following concepts:

- Length contraction in the special theory of relativity.
- Time dilation in the special theory of relativity.
- Gauge-invariance in the context of electromagnetic fields.

(c) Define in words and in detail the concept of ”threshold energy” in the context of particle collisions.

PROBLEM 3

(a) The CO_2 molecule is a linear molecule with two oxygen atoms at each side of a carbon atom, with respective masses m and M. The deviation from the equilibrium positions of each atom can be written as $\eta_i = \sum_\alpha \Delta_{i\alpha}\Theta_\alpha$, where $\theta_\alpha = \mathrm{Re}\{C_\alpha e^{-i\omega_\alpha t}\}$ are the normal coordinates. The subindex i represents the three different atoms. Firstly, write V_{ij} and T_{ij} on matrix form. Secondly, identify the eigenfrequencies ω_1, ω_s, and ω_a. Thirdly, find the cofactors $\Delta_{i\alpha}$, for $i = 1,2,3$, $\alpha = 1,s,a$ by using the normalization condition $\sum_{i,j=1}^{3} T_{ij}\Delta_{i\alpha}\Delta_{j\beta} = \delta_{\alpha\beta}$ and the equation of motion $\sum_{j=1}^{3}(V_{ij} - \omega_\alpha^2 T_{ij})\Delta_{j\alpha} = 0$.

Consider now a different problem.

(b) The Hamiltonian for a particle with mass m is in cylindrical coordinates (r,θ,z) given by:

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_z^2}{2m} + V. \tag{2}$$

Assume that the potential is separable in the following way: $V = a(r) + b(z)$, where $a(r)$ and $b(z)$ are known functions. Put $S(q,\alpha,t) = W(q,\alpha) - \alpha t$ into the Hamilton-Jacobi equation:

$$H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0. \tag{3}$$

Assume now that W is separable in the following manner: $W = p_\theta \theta + S_1(r) + S_2(z)$, and show that the quantity

$$\beta = b(z) + \frac{1}{2m}[S_2'(z)]^2 \tag{4}$$

has to be a constant. Finally, write down the solution for Hamilton's principal function on integral form.

PROBLEM 4

(a) Two point masses m are joined by a rigid weightless rod of length $l/2$, the center of which is constrained to move on a circle of radius r_0. Assume that the masses are also restricted to move in the plane defined by the motion of the center of the rod. Derive the total kinetic energy of this system expressed in terms of two generalized coordinates.

(b) Define in words, and in detail, what the differential scattering cross section gives information about physically. Also define in words the difference between the differential scattering cross section and the total scattering cross section.

(c) Explain in detail the relation between symmetries of the Lagrangian describing a given system and the possibility of having conserved quantities. Give at least three examples of important symmetries a Lagrangian can have and the corresponding conserved physical quantities.

(d) Describe how frictional forces may be included in Lagrange's equations via the Rayleigh dissipation function and provide the typical form for this function.

Supplementary Information

The regime of validity and the meaning of the symbols below are assumed to be known by the reader.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}. \tag{5}$$

$$[u,v]_{q,p} = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right) \tag{6}$$

$$\begin{aligned} x_\mu &= (\mathbf{r}, \mathrm{i}ct), \\ p_\mu &= (\mathbf{p}, \mathrm{i}E/c) \end{aligned} \tag{7}$$

$$A_\mu = (\mathbf{A}, \mathrm{i}\phi/c),\ \mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t,\ \mathbf{B} = \nabla\times\mathbf{A} \tag{8}$$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{9}$$

From the above equations, it follows that the general form of $F_{\mu\nu}$ in a given reference system is:

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -\mathrm{i}E_x/c \\ -B_z & 0 & B_x & -\mathrm{i}E_y/c \\ B_y & -B_x & 0 & -\mathrm{i}E_z/c \\ \mathrm{i}E_x/c & \mathrm{i}E_y/c & \mathrm{i}E_z/c & 0 \end{pmatrix} \tag{10}$$

$$F'_{\mu\nu} = L_{\mu\alpha}L_{\nu\beta}F_{\alpha\beta}. \tag{11}$$

The Lorentz-transformation matrix for the situation in Fig. 3 is given by:

$$L_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \mathrm{i}\beta\gamma \\ 0 & 0 & -\mathrm{i}\beta\gamma & \gamma \end{pmatrix} \tag{12}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

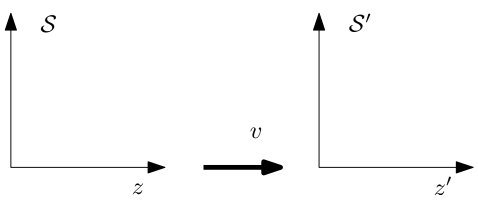

FIG. 3: Lorentz-transformation along the z-axis.