



NTNU – Trondheim
Norwegian University of
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Department of (name)

Examination paper for (TFY4345) (Classical Mechanics)

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Permitted examination support material:

- **Approved, simple calculator**
- **K. Rottmann (matematisk formelsamling)**
- **Barnett & Cronin: Mathematical Formulae**

Other information: Grading: Problem 1 (30%), Problem 2 (35%), Problem 3 (20%), Problem 4 (15%)

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Informasjon om trykking av eksamensoppgave

Originalen er:

1-sidig ☐ **2-sidig** ☐

sort/hvit ☐ **farger** ☐

Checked by:

Problem 1 **Mass on a rolling wheel** **(30 %)**

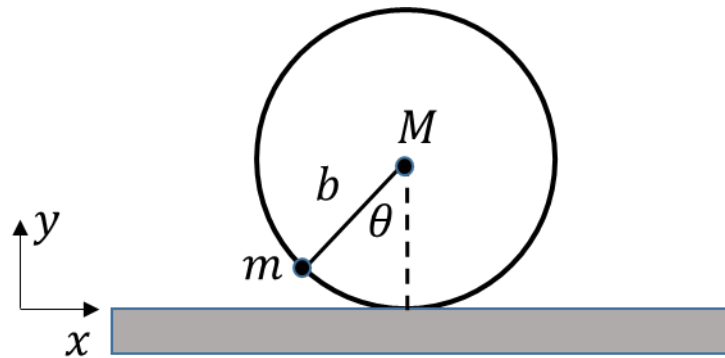


Figure 1. Mass m attached to wheel rolling on a surface (no slip).

A mass m is attached to the perimeter of wheel of radius b . The wheel is massless, except for a point mass M at its center. The radius of the wheel is b . The system is subject to a uniform gravitational field: $\vec{g} = -g \vec{e}_y$. When $\theta = 0$ the position of the mass m is $x = 0, y = 0$ in Cartesian coordinates.

(1a) The velocity of the mass m is $\vec{v}$, and the velocity of the mass M is $\vec{V}$. Show that:

$$v^2 = 2b^2\dot{\theta}^2 [1 - \cos(\theta)]$$

$$V^2 = b^2\dot{\theta}^2$$

where $\dot{\theta}$ denotes the time derivative of the angle.

(1b) Write down the Lagrangian for this system.

(1c) Find the equation of motion for the angle θ .

(1d) Linearize the equation of motion (valid for small θ) and show that the wheel can oscillate. Calculate the frequency of oscillation as a function of m, M, b and g .

Problem 2 Central force problem producing a lemniscate orbit (35 %)

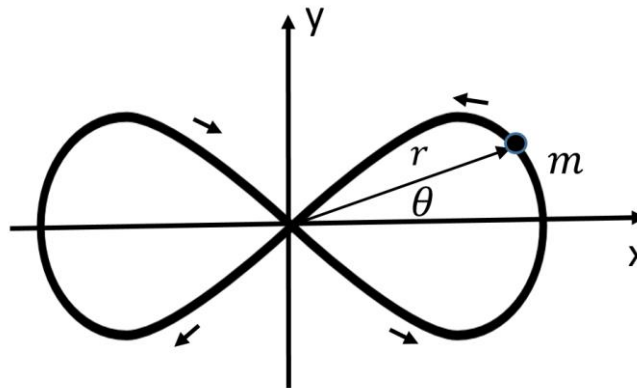


Figure 2. Particle moving in periodic orbit shaped like a “lemniscate curve”.

A particle of mass m is subject to an attractive central force potential:

$$V = -\frac{K}{r^6}$$

where r is the distance between the particle and the force centre ($x=0, y=0$).

(2a) Write down the Lagrangian for this system in polar coordinates r and θ .

(2b) Determine the conservation law associated with this Lagrangian. What is the conserved quantity?

(2c) Derive the equation of motion of the particle.

(2d) Show that there exist a solution of the form:

$$r = c \sqrt{\cos(2\theta)}$$

Derive an expression for c as a function of K , m and the conserved quantity calculated in question (2b). What is the total energy of the particle?

(2e) Solve the differential equation for $\theta(t)$ when $0 < \theta < \frac{\pi}{4}$. The initial condition is $\theta(t = 0) = 0$.

(2f) Calculate the orbital period as a function of m, K and c .

Problem 3 Precession of a frisbee (20 %)

(3a) Consider an axi-symmetric body with moment of inertia: $I_1 = I_2 \neq I_3$. The angular momentum in the laboratory frame is $\vec{L} = L\vec{e}_z$. Derive the equation of motion for the body, using the Euler equation and the angles θ, ψ, φ .

(3b) Find the expression for the Euler angles θ, ψ, φ as a function of time.

(3c) For a Frisbee $I_1 = I_2$ and $I_3 = 2 I_1$. The precession (wobble) of the frisbee is given by $\dot{\varphi}$. Show that the precession of the frisbee is twice as fast as the rotation frequency of the frisbee, assuming that the angle θ is small (*i.e.* $\cos(\theta) \approx 1$).

Problem 4. Special relativity: length contraction and rotation (15%)

(4a) A rod is moving along the z-axis with a uniform speed v . Assume first that the rod is parallel with the z-axis. The length of the rod in the moving reference system (x', y', z') is L' . Use the Lorentz transformations to calculate the length L of the rod in the stationary reference system (x, y, z) .

(4b) Assume now that the same rod is rotated so that it makes an angle θ_0 with the z' -axis. Calculate the length of the rod L as seen from the stationary reference system.

(4c) Show that the angle the rod makes with the z-axis is given by $\tan(\theta) = \frac{\tan(\theta_0)}{\sqrt{1 - \frac{v^2}{c^2}}}$. In other words the rod appears to have rotated.

Useful Formula and Equations

Rotation around Euler angle φ (rotation around z-axis):

$$D = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation around Euler angle θ :

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Rotation around Euler angle ψ :

$$B = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation from laboratory frame to body frame by Euler angle rotations:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = BCD \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Angular velocities in body frame:

$$\begin{aligned} \omega_{x'} &= \dot{\varphi} \sin(\theta) \sin(\psi) + \dot{\theta} \cos(\psi) \\ \omega_{y'} &= \dot{\varphi} \sin(\theta) \cos(\psi) - \dot{\theta} \sin(\psi) \\ \omega_{z'} &= \dot{\varphi} \cos(\theta) + \dot{\psi} \end{aligned}$$

Euler equation for a free body (zero external torque) :

$$\left(\frac{d\vec{L}}{dt} \right)_{body} + \vec{\omega} \times \vec{L} = 0$$

The angular momentum in a body frame:

$$\begin{aligned} L_{x'} &= L \sin(\theta) \sin(\psi) \\ L_{y'} &= L \sin(\theta) \cos(\psi) \\ L_{z'} &= L \cos(\theta) \end{aligned}$$

where the angular momentum in the laboratory frame is : $\vec{L} = L\vec{e}_z$.

Moment of inertia when body frame (x', y', z') is aligned with the principal axis of the body:

$$\begin{aligned} L_{x'} &= I_1 \omega_{x'} \\ L_{y'} &= I_2 \omega_{y'} \\ L_{z'} &= I_3 \omega_{z'} \end{aligned}$$

Lorentz transforms between a reference system S' moving with a constant velocity v in the z-direction, with respect to a reference system S:

$$\begin{aligned}x' &= x \\y' &= y \\z' &= \gamma(z - vt) \\t' &= \gamma\left(t - \frac{vz}{c^2}\right)\end{aligned}$$

$$\begin{aligned}x &= x' \\y &= y' \\z &= \gamma(z' + vt') \\t &= \gamma\left(t' + \frac{vz'}{c^2}\right)\end{aligned}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$