Problem 1. Set of questions (12 p = 30%)

True (T) or false (F). Please explain your choice as half of the points depends on it:

i. Only a certain class of orthogonal transformations can be used for describing rotations of rigid bodies.

ii. Consider a cylinder with the principal components of inertia $I_1 = I_2 > I_3$. Claim: The rotation around the third principal axis (I_3) is stable against small perturbations.

iii. Consider a system of N particles in three dimensions and k holonomic constraint equations. Claim: One must solve $3N$ Lagrange equations to reach the full solution.

iv. Consider shooting a rifle bullet under the influence of the Coriolis effect (on Earth). First, you will shoot towards East, and then you will shoot towards West at the same location. The distance is the same. Claim: You can apply the same Coriolis correction in your rifle settings for the two cases.

v. The angular momentum $\vec{L}$ and the angular velocity $\vec{\omega}$ always point in the same direction.

vi. A molecule of N atoms, has $3N - 3$ vibrational modes.

Problem 2. Condition for circular orbits (6 p = 15%)

Consider a central force field with a general potential $V(r)$ and a corresponding force $f = -\frac{\partial V}{\partial r}$.

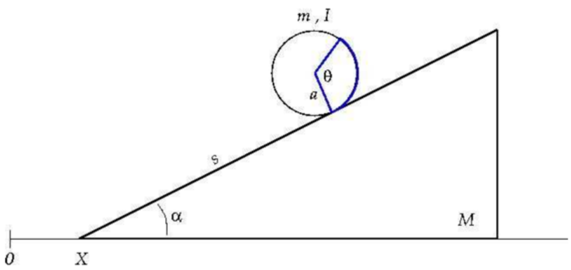
(a) Derive the Lagrangian function and solve the corresponding equations of motion. What are the conserved quantities? [2p]

(b) Solve the conditions for force $f(r_o)$ and total energy $E(r_o)$ in the case of circular orbits ($r = r_o$). Note that we have not specified $V(r)$ yet and that the results will remain somewhat implicit. For example, we cannot solve r_0 without knowing the potential explicitly. [1p]

(c) By considering the effective potential V_{eff} alone, deduce what is the condition that ensures that a circular orbit ($r = r_o$) corresponds to a stable minimum. Next, let us assume that we have a potential $V(r) = kr^{n+1}$, where n is an integer. Derive the stability condition of circular orbits for this set of power law potentials with respect to n. [3p]

Problem 3. Hoop rolling down an inclined plane (6 p = 15%)

A ring of mass m and radius R rolls down an inclined plane without slipping (see Figure). Let us denote its moment of inertia simply I. The inclined plane (mass M) has an opening angle α and it slides itself frictionlessly along a horizontal surface (instantaneous position X).



(a) Figure out the appropriate generalized coordinates and write out the Lagrangian function. [3p]

Hint: Start from the position coordinates for the center of the ring. Use the slipping constraint the same way as we are used to do with holonomic constraints (note: no forces requested).

(b) Solve the equations of motion and resulting accelerations for the ring and inclined plane. [3p]

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Problem 4. Hamiltonian mechanics and canonical transformations (7 p = 17.5%)

A particle of mass m which is described by one generalized coordinate q moves under the influence of a potential $V(q)$ and a damping force $f = -2m\gamma\dot{q}$ proportional to its velocity.

(a) Show that the following Lagrangian results in an equation of motion which physically corresponds to the above-mentioned starting situation. [2p]

$$L = e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)$$

(b) Obtain the Hamiltonian $H(q, p, t)$ for this system. [1p]

(c) Consider the following generating function: $F = e^{\gamma t}qP - QP$
Derive the basic equations for the canonical transformation of this type and obtain the transformation from (q, p) to (Q, P) and the transformed Hamiltonian $K(Q, P, t)$. [2p]

(d) Let us choose $V(q) = \frac{1}{2}m\omega^2 q^2$ as a harmonic potential with a natural frequency ω. Show that the transformed Hamiltonian $K(Q, P, t)$ is a constant of motion. [1p]

(e) Obtain the solution $Q(t)$ for the case $\gamma < \omega$ by solving Hamilton's equations in the transformed coordinates. Then, write down the solution $q(t)$ using the canonical transformation in (c). [1p]

Problem 5. Relativistic collision, Compton scattering (9 p = 22.5%)

A photon may be described classically as a particle of zero mass possessing nevertheless a momentum $p_c = h/\lambda = h\nu/c$, and therefore a kinetic energy $h\nu$. If the photon collides with an electron of mass m at rest, it will be scattered at some angle θ with a new energy $h\nu'$ and momentum h/λ'. The new relativistic linear momentum of the electron is p_e and its recoil angle is φ.

(a) What are the two conserved quantities? Derive the 4-vector in question from the event vector $\bar{\bar{x}} = [x, y, z, ict]$. [2p]

(b) Show that the change in energy is related to the scattering angle by the formula

$$\lambda' - \lambda = 2\lambda_C \sin^2\frac{\theta}{2}$$

where $\lambda_C = h/mc$ is known as the Compton wavelength. [4p]

Hint: Consider the conserved quantity in the xz-plane before and after the collision. You will get three equations. Make sure that p_e is visible in all of them. Useful trigonometric relation: $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$.

(c) Show also that the kinetic energy of the recoil motion of the electron is

$$T = h\nu\frac{2(\lambda_c/\lambda)\sin^2\frac{\theta}{2}}{1 + 2(\lambda_c/\lambda)\sin^2\frac{\theta}{2}}$$

Hint: This one is simpler than (b) and can be solved independently by using the given result from (b). [3p]