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Suggested solution for Exam TFY4345: Classical Mechanics

NOTE: The solutions below are meant as guidelines for how the problems may be solved and do not necessarily contain all the required details *e.g.* calculations.

PROBLEM 1

(a) Assume without loss of generality that the center of the rod moves in the xy-plane. Let θ be the angle representing how far around the circle the center of the rod has moved, and let ϕ be the angle that the rod makes with the x-axis. The position of the center of the rod is now $(x,y) = (r_0\cos\theta, r_0\sin\theta)$. Therefore, the positions of the masses relative the center of the rod will be:

$$(x_r, y_r) \pm \frac{1}{2}(l\cos\phi, l\sin\phi). \tag{1}$$

The absolute position of the masses are then expressed as:

$$(x,y) = (r_0\cos\theta \pm \frac{l}{2}\cos\phi, r_0\sin\theta \pm \frac{l}{2}\sin\phi). \tag{2}$$

The velocities of the two masses are obtained by $(\dot{x},\dot{y})$ and the magnitude squared of this vector beomes:

$$|v|^2 = r_0^2\dot{\theta}^2 + \frac{l^2}{4}\dot{\phi}^2 \pm r_0 l\dot{\theta}\dot{\phi}\cos(\theta-\phi). \tag{3}$$

The total kinetic energy is now obtained by summing the kinetic energy of each particle, for which it is seen that the third term ($\pm$) cancels. Thus, the total kinetic energy beomes:

$$T = m(r_0^2\dot{\theta}^2 + \frac{l^2}{4}\dot{\phi}^2). \tag{4}$$

(b) The differential scattering cross section expresses the ratio of the number of particles scattered into a specific part of space per time and the intensity of the incident particles. The total scattering cross section is a measure for the effective scattering area that incident particles will be subject to.

(c) If a Lagrangian has a continuous symmetry, for instance invariance under time-translation $t \to t+\Delta t$ or space-translation $r \to r+\Delta r$, it also has a belonging conserved quantity. This is related to cyclic coordinates and corresponding conserved canonical momenta. Three concrete examples of such symmetries and belonging conserved quantities are rotational symmetry (angular momentum), translational symmetry (momentum), and time-independence (energy).

(d) See section 1.5.2 in the compendium.

(e) In the context of particle collisions, the threshold energy is the minimum kinetic energy required to enable a reaction.

(f) Length contraction refers to the fact that the length of a moving object as measured by an observer in a stationary system, where the object is moving, and an observer in the rest frame of the object will be different: the length of the object is measured to be shorter in the frame moving with respect to the object.

Time dilation refers to the fact that the time measured on a moving clock by an observer in a stationary system, where the object is moving, and an observer in the rest frame of the clock will be different: time runs slower in the rest frame of the clock.

Gauge-invariance refers to the fact that we have a freedom in choosing the electric scalar potential ϕ and the magnetic vector potential $\mathbf{A}$ for a given electric $\mathbf{E}$ and magnetic field $\mathbf{B}$. For instance, since $\nabla\times\mathbf{A} = \mathbf{B}$, we can always add a term $\nabla\xi$ to $\mathbf{A}$ and obtain the same physical field $\mathbf{B}$. To do so, we also have to change the electric scalar potential correspondingly to leave the electric field invariant as well.

PROBLEM 2

(a) The kinetic and potential energies of the particle may be written down as:

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) = \frac{m}{2}[\dot{r}^2 + r^2(\dot{\theta}^2 + 4\alpha^2\dot{r}^2)], \tag{5}$$

$$V = mgz = mg\alpha r^2. \tag{6}$$

The Lagrangian is:

$$L = \frac{m}{2}[(1+4\alpha^2 r^2)\dot{r}^2 + r^2\dot{\theta}^2] - mg\alpha r^2, \tag{7}$$

so that it is clear that θ is a cyclic coordinate. Hence, the angular momentum $l = mr^2\dot{\theta}$ is conserved and we may eliminate $\dot{\theta}$ from the problem so that it becomes effectively 1D in that it only depends on r. The Lagrange-equation for r is then obtained as:

$$m(1+4\alpha^2 r^2)\ddot{r} + 4m\alpha^2 r\dot{r}^2 - \frac{l^2}{mr^3} + 2mg\alpha r = 0. \tag{8}$$

This equation determines the time-dependence of $r = r(t)$. Now, if we have circular motion, it follows that r is a constant and hence $\ddot{r} = \dot{r} = 0$. In this case, the equation of motion becomes:

$$\frac{l^2}{mr^3} = 2mg\alpha r \tag{9}$$

which may alternatively be expressed via the angular frequency $\dot{\theta}$ as

$$\dot{\theta} = \sqrt{2g\alpha}. \tag{10}$$

(b) See page 103 in the Goldstein book for a complete derivation. **NB!** There is a typo in the exam, it should be $f(r) = -k/r^2$ (in effect, a minus sign is missing). For this reason, all students have been given full score on this particular problem.

(c) See section 5.7 in the compendium for a complete derivation.

PROBLEM 3

(a) See page 65 in the compendium for a complete derivation.

(b) See pages 82-83 in the compendium for a complete derivation. If we relaxed the assumption about a center-of-mass at rest, we would have found an additional solution $\omega = 0$, which corresponds to a uniform translational motion of the entire molecule.

(c) After carrying out the scalar product between the velocity vector and the magnetic vector potential and also inserting polar coordinates ($x = r\cos\theta, y = r\sin\theta$), one obtains the Lagrangian:

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{qB}{2c}r^2\dot{\theta} - \frac{k}{2}r^2. \tag{11}$$

We obtain the Hamiltonian via the usual prescription of introducing the canonical momenta:

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + \frac{qB}{2c}r^2. \tag{12}$$

We then have:

$$H = \frac{p_r^2}{2m} + \frac{1}{2mr^2}\left(p_\theta - \frac{qB}{2c}r^2\right)^2 + \frac{1}{2}kr^2, \tag{13}$$

and so we may identify the Hamilton-Jacobi equation as:

$$\frac{1}{2m}\partial_r S^2 + \frac{1}{2mr^2}\left(\partial_\theta S - \frac{qB}{2c}r^2\right)^2 + \frac{1}{2}kr^2 - \partial_t S = 0. \tag{14}$$

PROBLEM 4

(a) Within the rest frame of the pion, energy and momentum conservation dictates that:

$$m_\pi^2 c^2 = \sqrt{m_\mu^2 c^4 + p_\mu^2 c^2} + p_\nu c \tag{15}$$

$$0 = \mathbf{p}_\mu + \mathbf{p}_\nu, \tag{16}$$

such that the muon and neutrino must have equal and opposite momentum. Combining these two equations, we obtain the kinetic energy of the muon as:

$$K_\mu = E_\mu - m_\mu c^2 = \frac{(m_\pi - m_\mu)^2 c^2}{2m_\pi}. \tag{17}$$

(b) Assume the photon is travelling initially along the z-axis (there is no loss of generality in assuming this). After scattering, it makes an angle θ with the z-axis, while we can assume that the electron makes an angle ϕ. Now, construct the total four-momentum before and after the collision. Conservation of momentum and energy then gives us the following three equations:

$$(h/\lambda')\sin\theta + p_e \sin\phi = 0, \tag{18}$$

$$(h/\lambda')\cos\theta + p_e \cos\phi = h/\lambda, \tag{19}$$

$$h/\lambda' + \sqrt{m^2c^2 + p_e^2} = mc + h/\lambda. \tag{20}$$

By eliminating ϕ from this equation, it is possible to express the scattering angle θ in terms of the wavelength before and after the collision in the announced way. To find the kinetic energy of the electron, we rewrite the above equation expressing energy conservation in a slightly different form:

$$h/\lambda' + \gamma mc = mc + h/\lambda. \tag{21}$$

Now, the kinetic energy of the electron is $T = (\gamma - 1)mc^2$. Using the above equation to eliminate γ in this expression, one arrives at the announced result.

It is seen that in the special case of $\theta = 0$, the energy change of the photon is zero. In addition, the kinetic energy acquired by the electron is zero. In effect, no scattering has taken place: this is the only way to satisfy the conservation laws when all the motion occurs along one axis.