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Suggested solution for Exam TFY4345: Classical Mechanics

NOTE: The solutions below are meant as guidelines for how the problems may be solved and do not necessarily contain all the detailed steps of the calculations.

PROBLEM 1

(a) See compendium http://folk.ntnu.no/jacobrun/spring2011/comp.pdf pages 47-48.

(b) See compendium http://folk.ntnu.no/jacobrun/spring2011/comp.pdf pages 90-91 and 95.

(c) See compendium http://folk.ntnu.no/jacobrun/spring2011/comp.pdf pages 25-29.

(d) See compendium for the basic postulates. As for the particle collision, we use conservation of 4-momentum to solve this problem. We have that:

$$P_\mu = p_{1\mu} + p_{2\mu} \tag{1}$$

where $p_{i\mu}$ is the 4-momentum of the two colliding particles (with velocities v and $-v$ since it is the center of mass frame) and P_μ is the 4-momentum of the created particle m'. The particle m' is at rest so we obtain:

$$(0, iE/c) = (\mathbf{p}_1 + \mathbf{p}_2, \mathrm{i}E_1/c + \mathrm{i}E_2/c). \tag{2}$$

It follows from conservation of energy (the fourth component of the 4-vector) that:

$$2\gamma mc^2 = m'c^2, \tag{3}$$

since $E = m'c^2$ and $E_i = \gamma mc^2$. Therefore, $m' = 2\gamma m$ where $\gamma = 1/\sqrt{1-(v/c)^2}$.

PROBLEM 2

(a) Energy conservation dictates that $E = m\dot{x}^2/2 + \gamma x^4/4$. The turning points in the motion occur when $\dot{x} = 0$ which we assume happens at $x = \pm x_0$, so that $E = \gamma x_0^4/4$. The period of the motion T is then equal to 4 times the time T_{x_0} required for the system to go from $x = 0$ to $x = x_0$:

$$\begin{aligned} T &= 4\int_0^{T_{x_0}} dt = 4\int_0^{x_0} \frac{dx}{\sqrt{2(E-V)/m}} \\ &= 4\sqrt{2m/\gamma}\frac{1}{x_0^2}\int_0^{x_0} \frac{dx}{\sqrt{1-(x/x_0)^4}} \\ &\simeq 7.42\sqrt{m/\gamma}\Big(\frac{\gamma}{4E}\Big)^{1/4} \end{aligned} \tag{4}$$

after performing the integral (listed in the Supplementary Information).

(b) Obtaining the equations of motion for the Lagrangian yields:

$$\ddot{x} + \frac{\dot{h}}{h}\dot{x} + \omega^2 x = 0, \tag{5}$$

so that $\dot{h}/h = \lambda$ must be satisfied. With initial condition $h(0) = 1$, the solution is $h(t) = \mathrm{e}^{\lambda t}$.

(c) To construct the Hamiltonian $H = \dot{x}p - L$, we must eliminate $\dot{x}$ in favor of p according to $p = \partial L/\partial \dot{x}$. We obtain $p = h(t)m\dot{x}$, so that:

$$H = \mathrm{e}^{-\lambda t}\frac{p^2}{2m} + \mathrm{e}^{\lambda t}\frac{m\omega^2 x^2}{2}. \tag{6}$$

This Hamiltonian has an explicit time-dependence and is therefore not conserved.

(d) The transformation equations dictated by the generating function F_2 are listed in the Supplementary Information:

$$p = \partial F_2/\partial x,\ X = \partial F_2/\partial P,\ K = H + \partial F_2/\partial t. \tag{7}$$

Using these equations, we find an expression for p and x in terms of P and X, which provides us with:

$$K = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2} + \gamma XP. \tag{8}$$

There is no explicit time-dependence here, and the Hamiltonian is conserved.

PROBLEM 3

(a) The moment of inertia around the pivot point where the rod is attached may be written as the sum of the moment of inertia due to the rod itself and the snail:

$$I = I_{\text{rod}} + I_{\text{snail}} = ML^2/3 + ml^2. \tag{9}$$

The Lagrangian $L = T - V$ is then:

$$L = I\dot{\theta}^2/2 + mv^2/2 - (-Mg\cos\theta L/2 - mg\cos\theta l). \tag{10}$$

We have defined θ as the angle between the rod and the line defined by the rod hanging at rest (vertically, meaning $\theta = 0$). Since v is a constant, the second term above may simply be discarded. For the potential energy, we have chosen the reference level ($V = 0$) to be at the height of the pivot $\theta = \pi/2$. We then obtain the Lagrange-equations via:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}. \tag{11}$$

Inserting our L, we obtain:

$$I\ddot{\theta} + \frac{dI}{dt}\dot{\theta} = \sin\theta(-MgL/2 - mgl). \tag{12}$$

Note that the moment of inertia I is time-dependent here since the position of the snail on the rod is time-dependent. We find:

$$dI/dt = 2ml\dot{l}, \tag{13}$$

where $\dot{l} = v$. Thus, the final result for the equation of motion is:

$$(ML^2/3 + ml^2)\ddot{\theta} + 2mlv\dot{\theta} + \sin\theta g(ML/2 + ml) = 0. \tag{14}$$

(b) Let $\eta_{1,2}$ denote the displacement of the masses from their equilibrium position. The Lagrangian then reads:

$$L = \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2) - \frac{1}{2}[k\eta_1^2 + k\eta_2^2 + 3k(\eta_1 - \eta_2)^2] \tag{15}$$

The first two terms in the potential energy part correspond to the potential energy due to the stretching/compression of the two springs with constants k. The third term in the potential energy part corresponds to the potential energy due to stretching/compression of the spring with constant $3k$, which depends on the *relative* displacement $\eta_1 - \eta_2$ between the masses. The eigenfrequences are computed as the solution of the equation $\det(\hat{V} - \omega^2\hat{T}) = 0$, where $\hat{V}$ and $\hat{T}$ are the potential and kinetic energy matrices. The elements of these matrices are identified from the form of the Lagrangian, and due to the cross-product terms $\sim \eta_1\eta_2$ in L, $\hat{V}$ will have off-diagonal terms. We find that

$$\hat{V} - \omega^2\hat{T} = \begin{pmatrix} 4k - \omega^2 m & -3k \\ -3k & 4k - \omega^2 m \end{pmatrix} \tag{16}$$

so that the determinant equation gives two solutions:

$$\omega = \sqrt{k/m} \text{ and } \omega = \sqrt{7k/m}. \tag{17}$$