Classical Mechanics TFY4345 - Exam 2015

1.

1a)

Particle position:

$$\vec{r} = \ell cos\theta \vec{e}_x + \ell sin\theta \vec{e}_y \tag{1}$$

Velocity:

$$\vec{v}^2 = \left(\frac{d\vec{r}}{dt}\right)^2 \tag{2}$$
$$= \dot{\ell}^2 + \ell^2\dot{\theta}^2 \tag{3}$$
$$= \dot{\ell}^2 + \ell^2\omega^2 \tag{4}$$

Lagrangian:

$$L = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\omega^2\right) \tag{5}$$

1b)

Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\ell}} = \frac{\partial L}{\partial l} \tag{6}$$

which implies:

$$\ddot{\ell} = \omega^2 \ell \tag{7}$$

The solution is:

$$\ell = Ae^{\omega t} + Be^{-\omega t} \tag{8}$$

With initial conditions $\ell(0) = \ell_0$ and $\dot{\ell}(0) = 0$ the solution is:

$$\ell(t) = \frac{1}{2}\ell_0\left(e^{\omega t} + e^{-\omega t}\right) = \ell_0 \cosh \omega t \tag{9}$$

1c) Lagrangian:

$$L = T - V = \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\omega^2\right) - mg\ell \sin \omega t \tag{10}$$

Lagrange equation:

$$\ddot{\ell} = \omega^2\ell - g \sin \omega t \tag{11}$$

1d)

$$\ell(t) = \frac{1}{2}\left(\ell_0 - \frac{g}{2\omega^2}\right)e^{\omega t} + \frac{1}{2}\left(\ell_0 + \frac{g}{2\omega^2}\right)e^{-\omega t} + \frac{g}{2\omega^2}\sin \omega t \tag{12}$$

1e)

If $\left(\ell_0 - \frac{g}{2\omega^2}\right) < 0$ we must have that $\ell(t) \to -\infty$ when $t \to \infty$. I.e. when $\omega^2 < \frac{g}{2\ell_0}$ $\ell(t)$ must become equal to $l = 0$ at some time t.

2.

2a)

Velocity:

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{13}$$

Lagrangian:

$$L = T - V = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}Kr^2 \tag{14}$$

2b) The Lagrangian does not depend on θ, which gives the conservation law:

$$\frac{\partial L}{\partial \dot{\theta}} = const. \tag{15}$$

Setting in L gives:

$$mr^2\dot{\theta} = \ell \tag{16}$$

where we call the constant ℓ (not to be confused with the length in Problem 1). ℓ is the angular momentum, similarly to the Kepler problem treated in the lectures and Brevik compendium.
2c)

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}Kr^2 \tag{17}$$

Inserting the equation for angular momentum conservation gives:

$$e = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + \frac{K}{2}r^2 \tag{18}$$

Solving for $\dot{r}$ and using the angular momentum law once more gives:

$$\frac{d\theta}{dr} = \frac{\ell}{mr^2\sqrt{\frac{2E}{m} - \frac{\ell^2}{m^2r^2} - \frac{K}{m}r^2}} \tag{19}$$

2d)
Substituting $u = 1/r^2$ gives:

$$d\theta = -\frac{\ell}{2}\frac{du}{\sqrt{2Emu - \ell^2u^2 - Km}} \tag{20}$$

Using the integral given in the problem we find:

$$(\theta - \theta_0) = -\frac{1}{2}\arccos\left[-\frac{Em - \ell^2 u}{\sqrt{E^2m^2 - \ell^2mK}}\right] + \frac{1}{2}\arccos\left[-\frac{Em - \ell^2 u_0}{\sqrt{E^2m^2 - \ell^2mK}}\right] \tag{21}$$

This is equivalent to

$$\cos(2\theta + c) = \frac{Em - \ell^2 u}{\sqrt{E^2m^2 - \ell^2mK}} \tag{22}$$

where c is a constant. which gives

$$r^2 = \frac{\ell^2/(Em)}{1 + \sqrt{1 - \frac{\ell^2K}{E^2m}}\cos(2\theta + c)} \tag{23}$$

We may chose $\theta = 0$ as we like, setting $c = 0$ correspond to taking $r(\theta = 0) = r_min$.

$$r^2 = \frac{\ell^2/(Em)}{1 + \sqrt{1 - \frac{\ell^2K}{E^2m}}\cos(2\theta)} \tag{24}$$

i.e.

$$A = \ell^2/(Em) \tag{25}$$

$$B = \sqrt{1 - \frac{\ell^2K}{E^2m}} \tag{26}$$

2e) Inserting Eq (24) into $x^2/a^2 + y^2/b^2 = 1$ gives :

$$a^2 = \frac{A^2}{1+B} \tag{27}$$

$$b^2 = \frac{A^2}{1-B} \tag{28}$$

3.

3a)

$$\begin{bmatrix} L'_x \\ L'_y \\ L'_z \end{bmatrix} = BCD \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix} = BC \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix} = B \begin{bmatrix} 0 \\ L\sin\theta \\ L\cos\theta \end{bmatrix} = \begin{bmatrix} L\sin(\psi)\sin(\theta) \\ L\cos(\psi)\sin(\theta) \\ L\cos(\theta) \end{bmatrix} \tag{29}$$

3b)

Inserting the expressions for the angular velocity in the body frame (rotating body) into the equation of motion gives (components):

$$L\sin\psi\sin\theta = I_1\left[\dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi\right] \tag{30}$$

$$L\cos\psi\sin\theta = I_2\left[\dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi\right] \tag{31}$$

$$L\cos\theta = I_3\left[\dot\varphi\cos\theta + \dot\psi\right] \tag{32}$$

We may rewrite these equations on the following form:

$$L = I_1\left[\dot\varphi + \dot\theta\frac{\cos\psi}{\sin\psi}\right] \tag{33}$$

$$L = I_2\left[\dot\varphi - \dot\theta\frac{\sin\psi}{\cos\psi}\right] \tag{34}$$

$$L\cos\theta = I_3\left[\dot\varphi\cos\theta + \dot\psi\right] \tag{35}$$

When $I_1 = I_2$ we may substract the two first equations above, and obtain:

$$\dot\theta\left[\frac{\cos\psi}{\sin\psi} + \frac{\sin\psi}{\cos\psi}\right] = 0 \tag{36}$$

This implies

$$\dot\theta = 0 \tag{37}$$

The remaining equations now decouple, and the solution is simple:

$$\dot\varphi = \frac{L}{I_1} \tag{38}$$

$$\dot\psi = L\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\cos\theta \tag{39}$$

An alternative (and accepted) solution is to plug the suggested solution (problem text) into the equations of motion, and find the constants c_1, c_2, c_3.

3c)

$$(\omega_{x'}, \omega_{y'}) = (\dot{\varphi}\sin\theta\cos\psi, \dot{\varphi}\sin\theta\cos\psi) = \frac{L}{I_1}\sin\theta\left(\sin(\dot{\psi}t), \cos(\dot{\psi}t)\right) \tag{40}$$

which is a rotating vector with (constant) angular velocity $\Omega = \dot{\psi} = L\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\cos\theta$

3d) The Euler equation free body (no torque):

$$\left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega}\times\vec{L} = 0 \tag{41}$$

In the solution above (Eq. 37) $\theta = const.$, which implies that the time derivative in Eq. (41) is:

$$\left(\frac{d\vec{L}}{dt}\right)_{body} = \frac{d}{dt}\begin{bmatrix} L\sin(\psi)\sin(\theta) \\ L\cos(\psi)\sin(\theta) \\ L\cos(\theta)\end{bmatrix} = \begin{bmatrix} L\dot{\psi}\cos(\psi)\sin(\theta) \\ -L\dot{\psi}\sin(\psi)\sin(\theta) \\ 0\end{bmatrix} \tag{42}$$

The second term in the Euler equation Eq. (41) is:

$$\vec{\omega}\times\vec{L} = \begin{bmatrix} \omega_{y'}L_{z'} - \omega_{z'}L_{y'} \\ \omega_{z'}L_{x'} - \omega_{x'}L_{z'} \\ \omega_{x'}L_{y'} - \omega_{y'}L_{x'}\end{bmatrix} \tag{43}$$

The x' component of Eq. (43) is

$$\omega_{y'}L_{z'} - \omega_{z'}L_{y'} = \dot{\varphi}\sin(\theta)\sin(\psi)L\cos(\theta) - (\dot{\varphi}\cos(\theta) + \dot{\psi})L\sin(\theta)\cos(\psi) \tag{44}$$

$$= -L\dot{\psi}\cos(\psi)\sin(\theta) \tag{45}$$

The y'component:

$$\omega_{z'}L_{x'} - \omega_{x'}L_{z'} = L\dot{\psi}\sin(\psi)\sin(\theta) \tag{46}$$

The z' component is zero:

$$\omega_{x'}L_{y'} - \omega_{y'}L_{x'} = \dot{\varphi}\sin(\theta)\sin(\psi)\,L\sin(\theta)\cos(\psi) - \dot{\varphi}\sin(\theta)\cos(\psi)\,L\sin(\theta)\sin(\psi) \tag{47}$$

$$= 0 \tag{48}$$

This shows that the solution found in 3b) satisfies the Euler equation for a free body (as it should).

4.

4a)

We calculate $\frac{du_z}{dt}$ using the Lorenz transform, ie.

$$\frac{du_z}{dt} = \frac{dz' + vdt'}{dt' + \frac{v}{c^2}dz'} \tag{49}$$

which gives

$$u_x = \frac{u'_x}{\gamma(1 + \frac{vu'_z}{c^2})} \tag{50}$$

$$u_y = \frac{u'_y}{\gamma(1 + \frac{vu'_z}{c^2})} \tag{51}$$

$$u_z = \frac{u'_z + v}{1 + \frac{u'_z v}{c^2}} \tag{52}$$

$$\tag{53}$$

4b)

Let $u = u_z$. We have $u' = c/n$

$$u = \frac{c}{n}\frac{1+n\beta}{1+\beta/n} \tag{54}$$

which gives

$$u \approx \frac{c}{n} + v(1 - \frac{1}{n^2}) \tag{55}$$