

Classical Mechanics TIFY4345 - Exam 2016

1.

(1a)

Position of the center of the wheel:

$$X = b\theta \quad (1)$$

$$Y = b \quad (2)$$

Position of point mass m:

$$x = b\theta - b \sin \theta \quad (3)$$

$$y = b - b \cos \theta \quad (4)$$

Velocity of mass M:

$$V^2 = \dot{X}^2 + \dot{Y}^2 = b^2 \dot{\theta}^2 \quad (5)$$

Velocity of mass m:

$$v^2 = \dot{x}^2 + \dot{y}^2 = b^2 \dot{\theta}^2 (1 - \cos \theta)^2 + b^2 \dot{\theta}^2 \sin^2 \theta = 2b^2 \dot{\theta}^2 (1 - \cos \theta) \quad (6)$$

(1b)

Lagrangian:

$$L = \frac{1}{2}mv^2 + \frac{1}{2}MV^2 - mgy \quad (7)$$

$$= mb^2 \dot{\theta}^2 (1 - \cos \theta) + \frac{1}{2}Mb^2 \dot{\theta}^2 - mgb(1 - \cos \theta) \quad (8)$$

(1c)

Lagrange equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \quad (9)$$

Inserting equation (8) into (9) gives the equation of motion:

$$2mb\ddot{\theta}(1 - \cos \theta) + Mb\ddot{\theta} + mb\dot{\theta}^2 \sin \theta + mg \sin \theta = 0 \quad (10)$$

(1d)

Linearising equation (10) gives:

$$\ddot{\theta} + \frac{mg}{Mb} \theta = 0 \quad (11)$$

which gives harmonic oscillations with frequency:

$$\omega = \sqrt{\frac{mg}{Mb}} \quad (12)$$

2a)

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{K}{r^6} \quad (13)$$

2b) The Lagrangian does not depend on θ :

$$\frac{\partial L}{\partial \theta} = 0 \quad (14)$$

This implies the conservation law:

$$\frac{\partial L}{\partial \dot{\theta}} = \text{constant} \quad (15)$$

$$mr^2\dot{\theta} = \text{constant} = \ell \quad (16)$$

The conserved quantity is the angular momentum of the particle.

2c) We may derive the equation of motion from the conservation of angular momentum and from the Lagrange equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \quad (17)$$

It is however easier to derive the equation of motion from conservation of total energy (as in the Kepler problem). The total energy is:

$$E = T + V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{K}{r^6} \quad (18)$$

We can eliminate $\dot{\theta}$ by using Eq. (16) on conservation angular momentum conservation:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{K}{r^6} \quad (19)$$

Solving for \dot{r} and using the angular momentum law once more gives:

$$\frac{dr}{d\theta} = \frac{mr^2}{\ell} \sqrt{\frac{2E}{m} - \frac{\ell^2}{m^2r^2} + \frac{2K}{mr^6}} \quad (20)$$

2d) We may express the left term in Eq. (20) as $\frac{dr}{d\theta} = \frac{1}{2r} \frac{d}{d\theta}(r^2)$. Inserting $r^2 = c^2 \cos 2\theta$ in Eq. (20) we find after some algebra that:

$$(c^4\ell^2 - 2Km) - 2c^6mE \cos^3(2\theta) = 0 \quad (21)$$

This implies that $r = c\sqrt{\cos 2\theta}$ is a solution provided that

$$E = 0 \quad (22)$$

$$c^4 = \frac{2mK}{\ell^2} \quad (23)$$

i.e. the total energy is $E = 0$.

2e) Combining the solution $r = c\sqrt{\cos 2\theta}$ with the equation of conservation of angular momentum gives:

$$\ell = mr^2 \frac{d\theta}{dt} \Rightarrow \ell dt = mc^2 \cos(2\theta) d\theta \quad (24)$$

Integrating this equation gives

$$\ell t = mc^2 \sin(2\theta) \quad (25)$$

Which gives:

$$\theta(t) = \frac{1}{2} \arcsin\left(\frac{2\ell}{mc^2}t\right) \quad (26)$$

2f) At time $t = 0$ the particle starts in the position $r = c$ and $\theta = 0$. The particle moves towards the origin, corresponding to the position $r = 0$, which implies $\theta = \frac{\pi}{4}$. Therefore $0 < \theta < \frac{\pi}{4}$ corresponds to a quarter of a orbit (orbital period τ) This implies

$$\frac{2\ell}{mc^2} \frac{\tau}{4} = 1 \quad (27)$$

according to Eq. (26). The total orbital period is:

$$\tau = \frac{2mc^2}{\ell} = \frac{2m}{\ell^2} \sqrt{2mK} \quad (28)$$

3.

3a)

The Euler equation free body (no torque):

$$\left(\frac{d\vec{L}}{dt} \right)_{body} + \vec{\omega} \times \vec{L} = 0 \quad (29)$$

From Eq. (29) we find the Euler equation on component form:

$$I_1 \dot{\omega}_{x'} + \omega_{y'} \omega_{y'} (I_3 - I_2) = 0 \quad (30)$$

$$I_2 \dot{\omega}_{y'} + \omega_{x'} \omega_{z'} (I_1 - I_3) = 0 \quad (31)$$

$$I_2 \dot{\omega}_{z'} + \omega_{x'} \omega_{y'} (I_2 - I_1) = 0 \quad (32)$$

Angular velocities in body frame:

$$\omega_{x'} = \dot{\varphi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \quad (33)$$

$$\omega_{y'} = \dot{\varphi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \quad (34)$$

$$\omega_{z'} = \dot{\varphi} \cos \theta + \dot{\psi} \quad (35)$$

3b) When $I_1 = I_2$ Eq. (32) implies that $\omega_{z'} = \text{constant}$. This again implies that the angular momentum around z' axis body frame is constant, i.e. $L'_z = L \cos \theta = \text{constant}$, which implies

$$\theta = \text{constant} \quad (36)$$

, i.e. $\dot{\theta} = 0$. Eq. (30) and (31) can now be expressed as:

$$\dot{\omega}_{x'} = -\Omega \omega_{y'} \quad (37)$$

$$\dot{\omega}_{y'} = \Omega \omega_{x'} \quad (38)$$

with $\Omega = \frac{I_3 - I_1}{I_1} \omega_{z'} = \text{constant}$. This implies¹ that $\omega_{x'}^2 + \omega_{y'}^2 = \text{constant}$. Which in turn implies:

$$\omega_{x'}^2 + \omega_{y'}^2 = (\dot{\varphi} \sin \theta \sin \psi)^2 + (\dot{\varphi} \sin \theta \cos \psi)^2 = \dot{\varphi}^2 \sin^2 \theta = \text{constant} \quad (39)$$

We thus find that:

$$\dot{\varphi} = c_1 \quad (40)$$

where c_1 is a constant. Inserting Eq. (33) and (34) into Eq. (37) and (38) gives

$$\dot{\varphi} \dot{\psi} \sin \theta \cos \psi = -\Omega \dot{\varphi} \sin \theta \cos \psi \quad (41)$$

$$-\dot{\varphi} \dot{\psi} \sin \theta \sin \psi = \Omega \dot{\varphi} \sin \theta \sin \psi \quad (42)$$

This implies that

$$\dot{\psi} = -\Omega = -\frac{(I_3 - I_1)}{I_1} \omega'_{z'} = \left(\frac{1}{I_3} - \frac{1}{I_1} \right) L \cos \theta \quad (43)$$

¹Using Eq. (37) and (38) we find: $\frac{d}{dt} [\omega_{x'}^2 + \omega_{y'}^2] = 2\omega_{x'} \dot{\omega}_{x'} + 2\omega_{y'} \dot{\omega}_{y'} = -2\Omega \omega_{x'} \omega_{y'} + 2\Omega \omega_{y'} \omega_{x'} = 0$ which shows that $\omega_{x'}^2 + \omega_{y'}^2$ does not change with time.

Inserting Eq. (43) into Eq. (35) gives

$$\dot{\phi} = \frac{L}{I_1} \quad (44)$$

3c)

$$\frac{\dot{\phi}}{\omega'_z} = \frac{\dot{\phi}}{L \cos(\theta)/I_3} = \frac{L/I_1}{L \cos(\theta)/I_3} = \frac{I_3}{I_1 \cos \theta} \approx \frac{I_3}{I_1} = 2 \quad (45)$$

4a)

Using the Lorentz transformation for the endpoints of the rod gives:

$$z'_2 - z'_1 = \gamma(z_2 - vt) - \gamma(z_1 - vt) = \gamma(z_2 - z_1) \quad (46)$$

Which implies

$$L = \frac{L'}{\gamma} = L' \sqrt{1 - \frac{v^2}{c^2}} \quad (47)$$

4b)

$$L_z = \frac{L'_z}{\gamma} = \frac{L'}{\gamma} \cos \theta_0 \quad (48)$$

$$L_y = L'_y = L' \sin \theta_0 \quad (49)$$

$$L^2 = L_y^2 + L_z^2 = L'^2 \left(1 - \frac{v^2}{c^2} \cos^2 \theta_0 \right) \quad (50)$$

4c)

$$\frac{L'_y}{L'_z} = \frac{L_y}{\gamma L_z} = \tan \theta_0 \quad (51)$$

which implies:

$$\tan \theta = \frac{L_y}{L_z} = \gamma \tan \theta_0 = \frac{\tan \theta_0}{\sqrt{1 - v^2/c^2}} \quad (52)$$