

Classical Mechanics TFY4345 - Exam 2016

1. Motion of a particle confined to a surface

1a)

Position of the particle on the cylinder:

$$\vec{r} = R \cos(\phi) \vec{e}_x + R \sin(\phi) \vec{e}_y + z \vec{e}_z \quad (1)$$

Velocity:

$$\vec{v}^2 = \left(\frac{d\vec{r}}{dt} \right)^2 \quad (2)$$

$$= R^2 \dot{\phi}^2 + \dot{z}^2 \quad (3)$$

$$(4)$$

Potential energy:

$$V = mgz \quad (5)$$

Lagrangian:

$$L = T - V = \frac{1}{2}mv^2 - mgz = \frac{1}{2}m \left(R^2 \dot{\phi}^2 + \dot{z}^2 \right) - mgz \quad (6)$$

1b)

Lagrange equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \quad (7)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{z}} = \frac{\partial L}{\partial z} \quad (8)$$

which implies the conservation law :

$$\frac{d}{dt} \left(mR^2 \dot{\phi} \right) = mR^2 \ddot{\phi} = 0 \quad (9)$$

This implies that the angular velocity is a constant $\dot{\phi} = \omega_0 = \text{constant}$. Equation of motion for the z-coordinate:

$$m\ddot{z} = -mg \quad (10)$$

1c) At time = 0 the velocity in the z-direction is zero, and the total velocity is then $v = v_0 = \omega_0 R$, where ω_0 is constant. The solution for the ϕ and z coordinates:

$$\phi = \omega_0 t \quad (11)$$

$$z = z_0 - \frac{1}{2}gt^2 \quad (12)$$

This is a helice where the pitch increases with time.

$$x = R \cos(\omega_0 t) \quad (13)$$

$$y = R \sin(\omega_0 t) \quad (14)$$

$$z = z_0 - \frac{1}{2}gt^2 \quad (15)$$

1d) Spherical coordinates for a particle confined to the surface of a cone:

$$x = r \sin(\theta) \cos(\phi) \quad (16)$$

$$y = r \sin(\theta) \sin(\phi) \quad (17)$$

$$z = r \cos(\theta) \quad (18)$$

The motion of the particle on the cone can be described by these spherical coordinates setting $\theta = \alpha = \text{constant}$.

Lagrangian for the particle on the cone :

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V \quad (19)$$

$$= \frac{1}{2}m \left[\dot{r}^2 + r^2 \dot{\phi}^2 \sin^2(\alpha) \right] - mgr \cos(\alpha) \quad (20)$$

1e)

Lagrange equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \quad (21)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \quad (22)$$

Inserting the Lagrangian in equation (20) into equations (21) and (22) gives :

$$mr^2 \dot{\phi} = \text{constant} = \ell \quad (23)$$

$$m\ddot{r} = mr \dot{\phi}^2 \sin^2(\alpha) - mg \cos(\alpha) \quad (24)$$

1f) The angular frequency of the particle is $\omega = \dot{\phi}$, stable motion implies $\ddot{r} = 0$, and equation (24) implies:

$$\omega = \dot{\phi} = \sqrt{\frac{g \cos(\alpha)}{r \sin^2(\alpha)}} = \sqrt{\frac{g \cos(\alpha)}{R \sin(\alpha)}} \quad (25)$$

1g) Combining the conservation law and equation of motion gives:

$$\ddot{r} = -g \cos(\alpha) + \frac{\ell^2}{r^3 \sin^2(\alpha)} \quad (26)$$

We now look at small perturbation around the stable motion: $r = r_0 + \delta(t)$, where r_0 is the radius of the stable motion in question 1f), i.e. $r_0 = R/\sin(\alpha)$ Expanding to linear order in δ gives

$$\ddot{\delta} = -\frac{3\ell^2}{r_0^4 \sin^2(\alpha)} \delta = -3 \sin^2(\alpha) \omega^2 \delta \quad (27)$$

Which means that the frequency of the oscillatory motion is:

$$\Omega = \sqrt{3} \sin(\alpha) \omega \quad (28)$$

We therefore get that $\omega = \Omega$ when $\sin \alpha = \frac{1}{\sqrt{3}}$, i.e. $\alpha \approx 35, 2^\circ$

2. Particle in a central force potential

2a)

$$\frac{d}{dt}\vec{L} = \vec{N} \quad (29)$$

Central force implies zero torque:

$$\vec{N} = \vec{F} \times \vec{r} = -\nabla V(r) \times \vec{r} = -V'(r)\vec{e}_r \times \vec{r} = 0 \quad (30)$$

This implies $\frac{d}{dt}\vec{L} = 0$, i.e. conservation of angular momentum. Which means that the particle stays in the same plane.

2b) Velocity:

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2 \quad (31)$$

Lagrangian:

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{k}{r^4} \quad (32)$$

L does not depend on ϕ , which gives the conservation law

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = \text{constant} = \ell \quad (33)$$

This is the angular momentum. In addition the Lagrangian does not depend on time, which implies conservation of energy.

2c)

Lagrange equation for r coordinate:

$$m\ddot{r} = mr\dot{\phi}^2 - 4\frac{k}{r^5} \quad (34)$$

Using the conservation of angular momentum:

$$m\ddot{r} = mr\dot{\phi}^2 - 4\frac{k}{r^5} = \frac{\ell^2}{mr^3} - 4\frac{k}{r^5} \quad (35)$$

This is equivalent to a 1D problem:

$$m\ddot{r} = -\frac{dV_{\text{eff}}}{dr} \quad (36)$$

$$V_{\text{eff}} = \frac{\ell^2}{2mr^2} - \frac{k}{r^4} \quad (37)$$

2d)

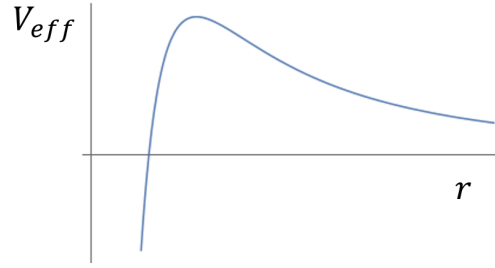


Figure 1: Sketch of potential $V_{\text{eff}} = \frac{\ell^2}{2mr^2} - \frac{k}{r^4}$ as a function of r . For small r the negative term $-\frac{k}{r^4}$ dominates, for large r the positive term $\frac{\ell^2}{2mr^2}$ dominates.

2e) $r = \text{constant}$ implies $\ddot{r} = 0$. Equation (36) hence implies $\frac{dV_{\text{eff}}}{dr}(r) = 0$, which has the solution $r_0 = \frac{2\sqrt{km}}{\ell}$.

2f) The orbits $r = r_0 = \text{constant}$ are unstable, since V_{eff} is maximum at r_0 . Any small displacement from r_0 will result in a gain potential energy, and the particle will move away from the orbit.

3. Cone rolling on a plane

3a) Velocity of centre of mass:

$$V_{\text{cm}} = \ell \cos(\alpha) \dot{\phi} \quad (38)$$

3b) Cone rotational motion is effectively a rotation around the instantaneous axis OA. The angular velocity around OA is

$$\omega = \frac{V_{\text{cm}}}{\ell \sin \phi} = \frac{\cos(\alpha)}{\sin(\alpha)} \dot{\phi} \quad (39)$$

3c) Let (x_1, x_2, x_3) be the coordinate system aligned with the principal axes of the cone. The projection of $\vec{\omega}$ on these axes is:

$$\vec{\omega} = \omega_1 \vec{e}_{x_1} + \omega_2 \vec{e}_{x_2} + \omega_3 \vec{e}_{x_3} = \omega \sin(\alpha) \vec{e}_{x_1} + \omega \cos(\alpha) \vec{e}_{x_3} \quad (40)$$

Component of ω along the x_3 axis:

$$\omega_3 = \omega \cos \alpha = \frac{\cos^2(\alpha)}{\sin(\alpha)} \dot{\phi} \quad (41)$$

3d)

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_3 \omega_3^2 \quad (42)$$

$$= \frac{1}{2} I_1 [\omega \sin(\alpha)]^2 + \frac{1}{2} I_3 [\omega \cos(\alpha)]^2 \quad (43)$$

$$= \frac{3}{40} H^2 m \dot{\phi}^2 [1 + 5 \cos^2(\alpha)] \quad (44)$$

4. Light from a fluorescent tube

4a)

In the S frame the tube lights up at the point z at time t . Seen from coordinate system S' :

$$z' = \gamma(z - vt) \quad (45)$$

$$t' = \gamma(t - \frac{vz}{c^2}) \quad (46)$$

In the S frame the tube lights up at the point $z + \Delta z$ at time t . Seen from coordinate system S' :

$$z' + \Delta z' = \gamma(z + \Delta z - vt) \quad (47)$$

$$t' + \Delta t' = \gamma(t - \frac{v(z + \Delta z)}{c^2}) \quad (48)$$

4b) From 4a) we get that:

$$\Delta z' = \gamma \Delta z \quad (49)$$

$$\Delta t' = -\frac{\gamma v \Delta z}{c^2} \quad (50)$$

Seen from S' the fluroscent tube does not light up instantaneously everywhere (like in S), the lighting up propagates with the velocity:

$$u = \frac{\Delta z'}{\Delta t'} = \frac{\gamma \Delta z}{-\frac{\gamma v \Delta z}{c^2}} = -\frac{c^2}{v} \quad (51)$$