

Exam 11.8.2018 - Solutions

1. a) $z' = \gamma(z - vt)$

$$L' = z'_2 - z'_1 = \gamma(z_2 - vt) - \gamma(z_1 - vt) = \gamma(z_2 - z_1)$$

$$\underline{L' = \gamma L}$$

Distance 494 km, $v = 0.8c$, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

$$\Rightarrow L = \frac{L'}{\gamma} = L' \sqrt{1 - v^2/c^2} = L' \cdot \frac{3}{5} = \underline{296 \text{ km}}$$

b) $\bar{x} = (x_1, x_2, x_3, ict)$; $\bar{v}^2 = \left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2$

$$dx_\mu = (dx_1, dx_2, dx_3, ic dt)$$

$$ds^2 = dx_\mu dx_\mu = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2 \quad (\text{invariant})$$

$$\overset{\text{rest frame}}{-c^2 d\tau^2} = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2 = (v^2 - c^2) dt^2$$

$$\Rightarrow dt = \frac{d\tau}{\sqrt{1 - \beta^2}} = \gamma d\tau$$

Velocity 4-vector:

$$u_\mu = \frac{dx_\mu}{d\tau} = \left(\frac{dx_i}{d\tau}, ic \frac{dt}{d\tau} \right) = \gamma(\bar{v}, ic)$$

$\uparrow$ proper time

Momentum 4-vector:

$$p_\mu = m u_\mu = (m\gamma\bar{v}, i p_4)$$

$$p_4 = \gamma mc$$

$$\text{Recall } T = \underbrace{\gamma mc^2}_E - mc^2 \quad \Leftrightarrow \quad E = T + mc^2 = \gamma mc^2$$

$\uparrow$ rest energy

$$\Rightarrow \underline{p_\mu = (m\gamma\bar{v}, i \frac{E}{c})}$$

2. Sliding pendulum

Holonomic constraints
(x-axis, rod)

a) 1° $U_1 = 0$, $T_1 = \frac{1}{2} m \dot{x}^2$

2° $U_2 = -m_2 g l \cos \theta$, $T_2 = \frac{1}{2} m_2 [\dot{x} \hat{i} + l \dot{\theta} (\hat{i} \cos \theta + \hat{j} \sin \theta)]^2$
 $= \dots = \frac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\theta}^2 + 2 \dot{x} l \dot{\theta} \cos \theta)$

$$L = T - U = \dots = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\theta}^2 + m_2 l \cos \theta (\dot{x} \dot{\theta} + g)$$

b) $x: \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{d}{dt} [(m_1 + m_2) \dot{x} + m_2 l \cos \theta \cdot \dot{\theta}] = 0$

$$\Rightarrow \underline{(m_1 + m_2) \ddot{x} + m_2 l \cos \theta \cdot \ddot{\theta} - m_2 l \sin \theta \cdot \dot{\theta}^2 = 0}$$

$\theta: -m_2 l \sin \theta (\ddot{x} \dot{\theta} + g) - \frac{d}{dt} (m_2 l^2 \dot{\theta} + m_2 l \cos \theta \cdot \dot{x}) = 0$

$$\Rightarrow \dots \Rightarrow \underline{m_2 g l \sin \theta + m_2 l^2 \ddot{\theta} + m_2 l \cos \theta \cdot \ddot{x} = 0}$$

c) $\dot{x} \rightarrow 0 \Rightarrow \ddot{x} \rightarrow 0$

The equation of motion for x becomes trivial.

$\theta: m_2 g l \sin \theta + m_2 l^2 \ddot{\theta} = 0$

$$\Rightarrow \underline{\ddot{\theta} + \frac{g}{l} \sin \theta = 0} \quad (\text{ordinary pendulum})$$

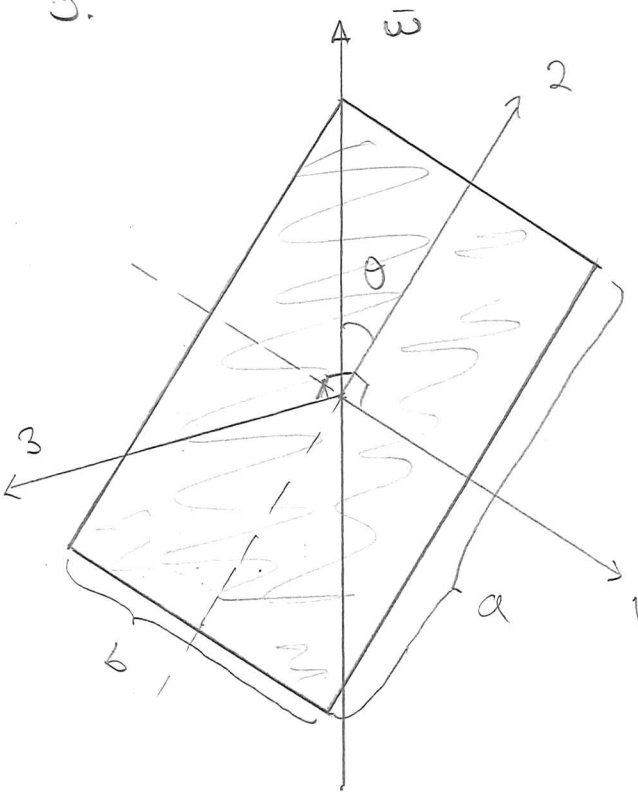
If θ small, $\sin \theta \approx \theta$

$$\Rightarrow \ddot{\theta} + \omega^2 \theta = 0, \quad \omega = \sqrt{\frac{g}{l}}$$

$$\theta(t) = \theta_0 \cos(\omega t)$$

($\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$)
init. conditions

3.

 $\bar{\omega}$ in 1-2 plane

$$I_1 = \frac{1}{12} m a^2$$

$$I_2 = \frac{1}{12} m b^2$$

$$I_3 = \frac{1}{12} m (a^2 + b^2)$$

tensor $\bar{I}$,
diagonal

$$\begin{aligned} \bar{L} &= \bar{I} \cdot \bar{\omega} = \omega_1 I_1 \hat{e}_1 + \omega_2 I_2 \hat{e}_2 + \omega_3 I_3 \hat{e}_3 \\ &= \frac{1}{12} m \omega [-\sin \theta \cdot a^2 \hat{e}_1 + \cos \theta \cdot b^2 \hat{e}_2] \end{aligned}$$

$$L^2 = \left(\frac{m \omega}{12} \right)^2 [\sin^2 \theta \cdot a^4 + \cos^2 \theta \cdot b^4]$$

$$\sin \theta = \frac{b}{\sqrt{a^2 + b^2}}, \quad \cos \theta = \frac{a}{\sqrt{a^2 + b^2}}$$

$$\begin{aligned} \Rightarrow L^2 &= \left(\frac{m \omega}{12} \right)^2 \left[\frac{b^2 a^4}{a^2 + b^2} + \frac{a^2 b^4}{a^2 + b^2} \right] \\ &= \left(\frac{m \omega}{12} \right)^2 a^2 b^2 \end{aligned}$$

$$\underline{L = \frac{1}{12} m \omega a b} \quad (*)$$

b) $\bar{\omega} \cdot \bar{L} = \omega L \cos \alpha$ (α unknown)

$$= \omega (-\sin \theta, \cos \theta) \cdot \frac{1}{12} m \omega (-\sin \theta \cdot a^2, \cos \theta \cdot b^2)$$

$$= \frac{1}{12} m \omega^2 (a^2 \sin^2 \theta + b^2 \cos^2 \theta) \quad (*)$$

$$= \frac{1}{12} m \omega^2 \left[\frac{b^2 a^2 + a^2 b^2}{a^2 + b^2} \right] = \frac{1}{12} m \omega a b \cdot \cos \alpha$$

$$\Rightarrow \cos \alpha = \frac{2ab}{a^2 + b^2} \Rightarrow \underline{\alpha = \arccos \frac{2ab}{a^2 + b^2}}$$

E.g. $b=1, a=2 \rightarrow \alpha = 36.7^\circ, \theta = 26.6^\circ$

c) $\underline{T_{\text{rot}} = \frac{1}{2} \bar{\omega} \cdot \bar{L} = \frac{1}{12} m \omega^2 \cdot \frac{a^2 b^2}{a^2 + b^2}}$

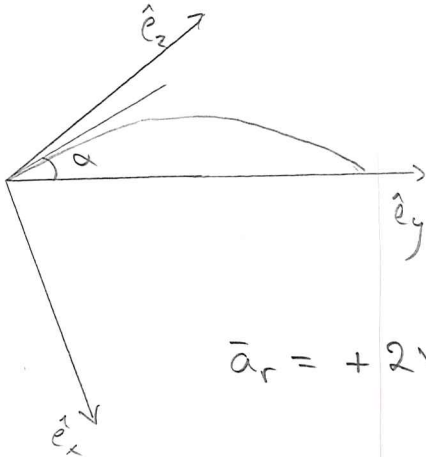
4. Coriolis effect

$$\bar{a}_r = \bar{g} - 2\bar{\omega} \times \bar{v}_r$$

$$\begin{cases} \omega_x = -\omega \cos \lambda \\ \omega_y = 0 \\ \omega_z = \omega \sin \lambda \end{cases} \quad \leftarrow \text{latitude}$$

$$\begin{cases} \dot{x} \approx 0 \\ \dot{y} = v_0 \cos \alpha \\ \dot{z} = v_0 \sin \alpha - gt \end{cases} \quad \leftarrow \text{shooting angle}$$

$$\omega \times \bar{v}_r = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -\omega \cos \lambda & 0 & \omega \sin \lambda \\ 0 & v_0 \cos \alpha & v_0 \sin \alpha - gt \end{vmatrix}$$



$$\text{Gravity: } \bar{g} = g \hat{e}_z$$

$$\bar{a}_r = +2v_0 \cos \alpha \cdot \omega \sin \lambda \hat{e}_x - 2(v_0 \sin \alpha - gt) \cdot \omega \cos \lambda \hat{e}_y + (g + 2v_0 \cos \alpha \cdot \omega \cos \lambda) \hat{e}_z$$

b) Neglect Coriolis in $\hat{e}_z$ -direction (vertical)

$$a_{r,x} = \frac{d^2 x}{dt^2} = +2\omega \sin \lambda \cdot v_0 \cos \alpha \quad \leftarrow \text{displacement}$$

$$\begin{aligned} \text{init. cond. } v_x=0 &\Rightarrow \frac{dx}{dt} = +2\omega \sin \lambda \cdot v_0 t \cos \alpha && \text{solve time} \\ \text{init. cond. } x=0 &\Rightarrow x(t) = +2\omega \sin \lambda \cdot \frac{1}{2} v_0 t^2 \cos \alpha = \omega \sin \lambda \cdot v_0 t^2 \cos \alpha \end{aligned}$$

$$a_{r,z} = \frac{d^2 z}{dt^2} = g + \underbrace{2v_0 \cos \alpha \cdot \omega \cos \lambda}_{\text{neglect this}}$$

$$\Rightarrow \dots \Rightarrow z(t) = \underbrace{z(0)}_{z_0} + \underbrace{v_0 \sin \alpha}_{\text{init. cond.}} \cdot t + \frac{1}{2} g t^2$$

$$\Rightarrow t = \frac{2v_0 \sin \alpha}{g}$$

$$\Rightarrow \underline{x(v_0, \alpha)} = +\omega \sin \lambda \cdot v_0 \cdot \left(\frac{2v_0 \sin \alpha}{g} \right)^2 \cos \alpha$$

$$= + \frac{4v_0^3}{g^2} \omega \sin \lambda \sin^2 \alpha \cos \alpha \quad (+ \leftrightarrow \text{South})$$

5. Hamiltonian dynamics

a) $x^2 + y^2 = R^2$

$$\vec{F} = -\frac{k}{r^2} = -\nabla U(r) \Rightarrow U = -\frac{k}{r} = -\frac{k}{\sqrt{R^2 + z^2}}$$

$$\vec{v}^2 = \dot{R}^2 + R^2 \dot{\theta}^2 + \dot{z}^2$$

$$T = \frac{1}{2} m (R^2 \dot{\theta}^2 + \dot{z}^2)$$

$$L = T - U = \frac{1}{2} m (R^2 \dot{\theta}^2 + \dot{z}^2) + \frac{k}{\sqrt{R^2 + z^2}} \quad (\text{no } \theta!)$$

$$\underline{p_\theta} = \frac{\partial L}{\partial \dot{\theta}} = \underline{m R^2 \dot{\theta}}, \quad \underline{p_z} = \frac{\partial L}{\partial \dot{z}} = \underline{m \dot{z}}$$

$$\underline{H(z, p_z, p_\theta)} = T + U = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} - \frac{k}{\sqrt{R^2 + z^2}}$$

b) θ cyclic coordinate $\rightarrow p_\theta$ conserved (angular momentum)
Total energy conserved $\nearrow$

a) (continues...)

$$\begin{cases} \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \\ \dot{p}_z = -\frac{\partial H}{\partial z} = +2zk \cdot \left(-\frac{1}{2}\right) \cdot (R^2 + z^2)^{-\frac{3}{2}} = -\frac{kz}{\sqrt{R^2 + z^2}^3} \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \\ \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \end{cases}$$

Hamilton's equation of motion

c) $\frac{dT}{dt} = [T, H]_{p,q} + \frac{\partial T}{\partial t}$; T time-independent $\Rightarrow \frac{\partial T}{\partial t} = 0$

$$[T, H]_{p,q} = \sum_i \left(\frac{\partial T}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial T}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$= \underbrace{\frac{\partial T}{\partial \theta} \frac{\partial H}{\partial p_\theta}}_0 - \underbrace{\frac{\partial T}{\partial p_\theta} \frac{\partial H}{\partial \theta}}_0 + \underbrace{\frac{\partial T}{\partial z} \frac{\partial H}{\partial p_z}}_0 - \frac{\partial T}{\partial p_z} \frac{\partial H}{\partial z}$$

$$= + \frac{p_z}{m} \cdot \frac{kz}{\sqrt{R^2 + z^2}^3} \Rightarrow \underline{\frac{dT}{dt} = \frac{p_z}{m} \cdot \frac{kz}{\sqrt{R^2 + z^2}^3}}$$

$$[T, H]_{p,q} = 0 \quad \text{if } z \text{ constant} \rightarrow p_z = 0$$

$\Rightarrow$ circular orbit