TFY4345 Summer 2019 Solutions

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Problem 1

(a)
i. **False**. The method of Lagrange's undetermined multiplier can also be used for systems that have non-holonomic constraints of the form $f(q, \dot{q}, t)$.
ii. **False**. When one of the events lies outside the light cone of the other, they can never be causally connected. This is due to the fact that there are no signal that can travel faster than the speed of light (special relativity).
iii. **False**. Kepler orbits are conical intersections, including circle, ellipse, parabola and hyperbola.
(b)
The angular momentum $\vec{L}$ is defined as,

$$\vec{L} = \vec{r} \times \vec{p} = \epsilon_{ijk} x_j p_k. \tag{1}$$

The formal definition of the Poisson brackets is,

$$[L_z, L_y] = \frac{\partial L_z}{\partial q_i}\frac{\partial L_y}{\partial p_i} - \frac{\partial L_z}{\partial p_i}\frac{\partial L_y}{\partial q_i}, \tag{2}$$

where we have used Einsteins summation convention (repeated indices are being summed over). If we consider a particle moving in a velocity independent potential we can take the generalized coordinates to be the usual Cartesian coordinates $(q_1, q_2, q_3) = (x, y, z)$, and then the canonical momentum simply equals the physical momentum. A straight forward calculation of the terms in Eq.(2) yields,

$$\frac{\partial L_z}{\partial x_i} = \frac{\partial}{\partial x_i}(xp_y - yp_x) = \delta_{ix}p_y - \delta_{iy}p_x, \tag{3}$$

$$\frac{\partial L_z}{\partial p_i} = \frac{\partial}{\partial p_i}(xp_y - yp_x) = \delta_{iy}x - \delta_{ix}y, \tag{4}$$

$$\frac{\partial L_y}{\partial x_i} = \frac{\partial}{\partial x_i}(zp_x - xp_z) = \delta_{iz}p_x - \delta_{ix}p_z, \tag{5}$$

$$\frac{\partial L_y}{\partial p_i} = \frac{\partial}{\partial p_i}(zp_x - xp_z) = \delta_{ix}z - \delta_{iz}x. \tag{6}$$

Substituting these expressions back into Eq.(2) we get,

$$
\begin{aligned}
&[L_z, L_y] = (\delta_{ix}p_y - \delta_{iy}p_x)(\delta_{ix}z - \delta_{iz}x) - (\delta_{iy}x - \delta_{ix}y)(\delta_{iz}p_x - \delta_{ix}p_z) \\
&= zp_y - yp_z = -L_x \qquad (7)
\end{aligned}
$$

c)
We will choose a coordinate system with the first axis pointing to the south, the second axis pointing to the east and the third axis pointing outwards. In this coordinate system we get the following expressions for the velocity of the car v and the angular velocity of the earth ω:

$$\vec{v} = (0, v, 0), \qquad (8)$$

$$\vec{\omega} = (-\omega\cos\lambda, 0, \omega\sin\lambda). \qquad (9)$$

From the definition of the Coriolis force $F_C = 2mv \times \omega$ we get,

$$F_C = (2mv\sin\lambda, 0, 2mv\cos\lambda). \qquad (10)$$

We can neglect the third component of this force, which is contributing to the support force from the road, since mg is certainly much greater. By inserting the numerical values into the expression above ($\omega = 7.29 \times 10^5$, m and v are given in the text) we get $F_C = 19.89N \approx 20N$ towards the south.

Problem 2

a) The generalized coordinates are x and θ. The pendulum coordinates is expressed in terms of the generalized coordinates as,

$$(x_1, y_1) = (a + x + l\sin\theta, -l\cos\theta). \qquad (11)$$

The kinetic and potential energy are given by,

$$
\begin{aligned}
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x_1}^2 + \dot{y_1}^2) = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x_1} + l\cos\theta\dot{\theta})^2 + l^2\dot{\theta}^2\sin^2\theta] \\
&= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\dot{x}\dot{\theta}, \qquad (12) \\
V &= \frac{1}{2}kx^2 - mgl\cos\theta. \qquad (13)
\end{aligned}
$$

The Lagrangian $\mathcal{L} = T_V$ is then given by,

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\dot{x}\dot{\theta} - \frac{1}{2}kx^2 + mgl\cos\theta. \qquad (14)$$

From the definition of canonical momentum $p_q = \frac{\partial\mathcal{L}}{\partial\dot{q}}$ and canonical force

$F_q = \frac{\partial \mathcal{L}}{\partial q}$ we get,

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = (M+m)\dot{x} + ml\cos\theta\dot{\theta}, \tag{15}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml\cos\theta\dot{x} + ml^2\dot{\theta}, \tag{16}$$

$$F_x = \frac{\partial \mathcal{L}}{\partial x} = -kx, \tag{17}$$

$$F_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = -ml\sin\theta\dot{x}\dot{\theta} - mgl\sin\theta. \tag{18}$$

b)
The equation of motion is given by the Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0. \tag{19}$$

It follows from Eq.(14) that the equations of motion are,

$$x : (M+m)\ddot{x} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = -kx \tag{20}$$

$$\theta : ml\cos\theta\ddot{x} + ml^2\ddot{\theta} = -mgl\sin\theta. \tag{21}$$

c)
For small oscillations we have $cos \approx 1$ and $sin \approx \theta$ and we may linearise the equations of motion (neglecting also $\theta\dot{\theta}^2$, since this was not mentioned explicitly there was no reduction for points if it was not included),

$$x : (M+m)\ddot{x} + ml\ddot{\theta} + kx = 0 \tag{22}$$

$$\theta : ml\ddot{x} + ml^2\ddot{\theta} + mgl\theta = 0. \tag{23}$$

We rewrite the equations above in terms of the new variables given in the text and get,

$$x : (1+\alpha)\ddot{u} + \alpha\ddot{\theta} + \omega_0^2 u = 0, \tag{24}$$

$$\theta : \ddot{u} + \ddot{\theta} + \omega_1^2\theta. \tag{25}$$

Problem 3

(a)
Since the slab is uniform its mass M is $M = A \times \rho = \frac{1}{2}ab\rho$. Let x_{CM} denote the x-component of the center of mass (CM). Using the definition of CM we find,

$$x_{cm} = \frac{1}{M}\int_0^a dx \int_0^{b(1-\frac{x}{a})} dy \rho x = \frac{\rho b}{M}\int_0^a dx (1 - \frac{x}{a})$$

$$= \frac{a^2 b\rho}{M}\int_0^1 du(1-u)u = \frac{\rho a^2 b}{6M} = \frac{a}{3}, \tag{26}$$

where we used the substitution $u = 1 - \frac{x}{a}$ which implies $dx = -adu$. Because of the geometry in the problem (the slab has a triangular shape) the calculation of y_{CM} is completely analogous, and the result is $y_{CM} = \frac{b}{3}$.

(b)
The slab is two dimensional and is laying in the xy-plane ($z = 0$). This implies that $I_{zx} = I_{xz} = I_{zy} = I_{yz} = 0$ and $I_{zz} = I_{xx} + I_{yy}$. All we need to calculate is then reduced to I_{xx},I_{yy} and $I_{xy} = I_{yx}$:

$$I_{xx} = \rho \int\limits_0^a dx \int\limits_0^{b(1-\frac{x}{a})} y^2 dy = \frac{\rho b^3}{3} \int\limits_0^a dx (1 - \frac{x}{3})^3 = \frac{\rho a b^3}{3} \int\limits_0^1 u^3 du = \frac{M}{6} b^2. \tag{27}$$

The computation of I_{yy} is completely analogous and the result is $I_{yy} = \frac{M}{6} a^2$. Finally we compute I_{xy},

$$I_{xy} = -\rho \int\limits_0^a dx \int\limits_0^{b(1-\frac{x}{a})} yx dy = -\frac{\rho b^2}{2} \int\limits_0^a x (1 - \frac{x}{a})^2 = -\frac{\rho a^2 b^2}{24} = -\frac{M}{12} ab. \tag{28}$$

Putting it all together we can write the inertia tensor on matrix form,

$$I = \frac{M}{6} \begin{pmatrix} b^2 & -\frac{1}{2}ab & 0 \\ -\frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} \tag{29}$$

(c)
By comparing the two matrices in the problem text we see that we can write the new variables as,

$$A = \frac{1}{2}(a^2 + b^2), B = \frac{1}{2}\sqrt{(b^2 - a^2)^2 + a^2 b^2}, \vartheta = \tan^{-1}\Big(\frac{ab}{b^2 - a^2}\Big). \tag{30}$$

The last equation describes a right triangle with side lengths $b^2 - a^2$, ab and $\sqrt{b^2 - a^2)^2 + a^2 b^2} = 2B$, where the angle ϑ is opposite to the side whose length is ab. From this observation we deduce that $ab = 2B\cos\vartheta$ and $b^2 - a^2 = 2B\cos\vartheta$. It follows that

$$a^2 = \frac{1}{2}(b^2 + a^2) - \frac{1}{2}(b^2 - a^2) = A - B\cos\vartheta, \tag{31}$$

$$b^2 = \frac{1}{2}(b^2 + a^2) + \frac{1}{2}(b^2 - a^2) = A + B\cos\vartheta, \tag{32}$$

and putting it all together we get,

$$I = \frac{M}{18} \begin{pmatrix} A + B\cos\vartheta & B\sin\vartheta & 0 \\ B\sin\vartheta & A - B\cos\vartheta & 0 \\ 0 & 0 & 2A \end{pmatrix} \tag{33}$$

To find the principal moments of inertia and the principal axis we first need to solve the characteristic equation $\det(I-\omega)=0$ for the principal moments of inertia ω. Thus we get,

$$\det(I-\omega)=(\omega-2A)[(B\cos\vartheta+A-\omega)(A-B\cos\vartheta-\omega)-B^2\sin^2\vartheta] \\ =(\omega-2A)[(\omega-A)^2-B^2]=0, \tag{34}$$

which has three solutions $\omega_1 = 2A$, $\omega_2 = A+B$ and $\omega_3 = A-B$. The corresponding principal axis are the corresponding eigenvectors V_i that satisfies $IV_i = \omega_i V_i$. The solutions are,

$$V_1 = (0,0,1), \tag{35}$$

$$V_2 = (\cos\frac{1}{2}\vartheta, \sin\frac{1}{2}\vartheta, 0), \tag{36}$$

$$V_3 = (-\sin\frac{1}{2}\vartheta, \cos\frac{1}{2}\vartheta, 0). \tag{37}$$

This is a general result for any right triangle. Note that V_1 points outside the xy-plane while V_2 and V_3 rotate as a function of a and b.

Problem 4

a)
We start by finding the potential V that is associated with the force F. From $F=-\nabla V$ it follows that $V(r)=\frac{1}{3}kr^3$. The velocity squared v^2 in cylinder coordinates is given by $v^2=\dot{R}^2+R^2\dot{\theta}^2+\dot{z}^2$, where $\dot{R}=0$ because the particle is constrained to move on the cylinder surface. The kinetic energy T is then $T=\frac{1}{2}m(R^2\dot{\theta}^2+\dot{z}^2)$, so that the Lagrangian becomes

$$L = T - V = \frac{1}{2}m(R^2\dot{\theta}^2+\dot{z}^2)-\frac{1}{3}kr^3, \tag{38}$$

where we notice that there is no θ dependence in the Lagrangian. The canonical momentum p_θ and p_z are,

$$p_\theta = \frac{\partial L}{\dot{\theta}} = mR^2\dot{\theta}, \tag{39}$$

$$p_z = \frac{\partial L}{\dot{z}} = m\dot{z}. \tag{40}$$

The Hamiltonian $H = H(z, \dot{\theta}, \dot{z})$ then becomes,

$$H = p_z\dot{z} + p_\theta\dot{\theta} - L = \frac{p_\theta^2}{2MR^2} + \frac{p_z^2}{2m} + \frac{1}{3}(R^2+z^2)^{\frac{3}{2}}. \tag{41}$$

Finally we find the Hamiltonian equations of motion,

$$p_\theta = -\frac{\partial H}{\partial \theta} = 0, \tag{42}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} = \text{constant}, \tag{43}$$

$$p_z = -\frac{\partial H}{\partial z} = -kz\sqrt{R^2 + z^2}, \tag{44}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}. \tag{45}$$

b)
The Euler-Lagrange equation of motion for the z coordinate becomes $m\ddot{z} + kz\sqrt{R^2 + z^2} = 0$.
Since θ is a cyclic coordinate it follows that p_θ (angular momentum) is conserved.
Obviously the total energy is also conserved.