PROBLEM 1 - Holonomic constrains and Poisson brackets

a. The definition of a holonomic constraint is *so* simple: $f_l(q_1, ..., q_n; t) = 0$, i.e. the constraint depends only on the generalized coordinates and possibly time. Together with the assumption of a conservative potential, this enables us to build the Lagrangian formalism from D'Alemberts principle (differential principle). However, it is possible to use the Lagrangian approach also in situations where the potential and constraints involve (generalized) velocities. This is related to Hamilton's principle (integral approach) that is valid for monogenic systems. A semiholonomic constraint that includes velocities can be written as $f_l(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n; t) = 0$, and in a more restricted form

$$f_l = \sum_{k=1}^{n} a_{l,k}\dot{q}_k + a_{l,t} = 0, \qquad l = 1, 2, ..., m \tag{1}$$

where we assume m contraint equations altogether. By multiplying by dt we achieve a differential from of the constraint

$$\sum_{k=1}^{n} a_{l,k}dq_k + a_{l,t}dt = 0, \qquad l = 1, 2, ..., m \tag{2}$$

which we have used in the case of a slipping contraint.

The extended Lagrange equation with undetermined multipliers becomes now

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = -\sum_{l=1}^{m} \lambda_l \frac{\partial f_l}{\partial \dot{q}_k} = -\sum_{l=1}^{m} \lambda_l a_{l,k} = Q_k, \qquad k = 1, 2, ..., n \tag{3}$$

The nice feature is that this equation can also be modified for holonomic constraints

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_{l=1}^{m} \lambda_l \frac{\partial f_l}{\partial q_k} = 0, \qquad k = 1, 2, ..., n \tag{4}$$

This formulation is particularly relevant for constraint equations with an inequality sign, and it enables us to calculate forces that are associated with the constraints (generalized forces denoted as Q_k). While using the method of Lagrange's undetermined multipliers, one has to keep the original n generalized coordinates and have m equations for the constraints in addition. Thereby, one starts the procedure with $n + m$ equations and unknowns.

b. The Hamiltonian is

$$H = T + V = \frac{p^2}{2m} + mgy \tag{5}$$

Obviously, we have a particle falling under a (constant) gravitational field. The total time derivative of kinetic energy is

$$\frac{dT}{dt} = [T, H]_{q,p} + \frac{\partial T}{\partial t} \tag{6}$$

Since there is no explicit time-dependence of T, the partial derivative is zero and it is enough to focus on the Poisson bracket. By definition, the relevant relation is

$$[T, H] = \left(\frac{\partial T}{\partial y} \frac{\partial H}{\partial p} - \frac{\partial H}{\partial y} \frac{\partial T}{\partial p} \right) \tag{7}$$

It is very easy to calculate each partial derivative

$$\frac{\partial H}{\partial p} = \frac{p}{m}, \quad \frac{\partial H}{\partial y} = mg, \quad \frac{\partial T}{\partial p} = \frac{p}{m}, \quad \frac{\partial T}{\partial y} = 0, \tag{8}$$

and the total derivative becomes

$$\frac{dT}{dt} = [T, H]_{q,p} = -gp \tag{9}$$

Alternatively, let us consider the fact that we are dealing with a conservative force field where $H = E$, i.e.

$$\frac{dE}{dt} = 0 = \frac{dT}{dt} + \frac{dV}{dt} \tag{10}$$

This leads to

$$\frac{dT}{dt} = -\frac{dV}{dt} = -mg\dot{y} = -gp \tag{11}$$

Remember that $p = m\dot{y}$! Note also the negative sign the result.

PROBLEM 2 - Sliding particle with air resistance

a. The Lagrangian for frictionless sliding with air resistance is

$$L = \frac{1}{2}m\dot{x}^2 - mg(\ell - x)\sin\theta \tag{12}$$

Where ℓ is the starting distance from the beginning of the slope. The Lagrange equation with external force becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = -km\dot{x} \tag{13}$$

The resulting equation of motion becomes

$$m\ddot{x} - mg\sin\theta = -km\dot{x} \tag{14}$$

This is enough at this stage and rather than solving x, we shall focus on velocity $\dot{x} = v$ next.

b. The corresponding equation for v becomes

$$\frac{dv}{dt} - g\sin\theta = -kmv \quad \Longrightarrow \quad \frac{dv}{kv - g\sin\theta} = -dt \tag{15}$$

Let us integrate on both sides

$$\int \frac{dv}{kv - g\sin\theta} = -t + C' \quad \Longrightarrow \quad \frac{1}{k}\ln(kv - g\sin\theta) = -t + C' \tag{16}$$

$$\Longrightarrow \quad kv - g\sin\theta = e^{-kt+C} \quad \Longrightarrow \quad v = \frac{g\sin\theta}{k} + \frac{e^{-kt+C}}{k} \tag{17}$$

At time $t = 0$, velocity is zero $v = v_0 = 0$, which leads to $e^C = -g\sin\theta$ and the final equation for velocity

$$v = \frac{g\sin\theta}{k}\left(1 - e^{-kt}\right) \tag{18}$$

The upper limit at $t \to \infty$ is $v = g\sin\theta/k$ and called terminal velocity.

c. The particle reaches 90% of the terminal velocity

$$v = \frac{g\sin\theta}{k}\left(1 - e^{-kt}\right) = 0.9 \times \frac{g\sin\theta}{k} \tag{19}$$

$$\implies \quad 1 - e^{-kt} = 0.9 \quad \implies \quad t = -\frac{\ln(0.1)}{k} \tag{20}$$

Let us solve x next via another integration

$$x = \int v dt = \frac{gt \sin\theta}{k} + \frac{g \sin\theta}{k^2} e^{-kt} + D \tag{21}$$

In the beginning $x = 0 \implies D = -\frac{g \sin\theta}{k^2}$ and

$$x = \frac{gt \sin\theta}{k} + \frac{g \sin\theta}{k^2}\left(e^{-kt} - 1\right) \tag{22}$$

Let us now insert the time

$$x = \frac{g \sin\theta}{k^2}\Big(\ln(10) - 0.9\Big) \tag{23}$$

Fun fact: Consider $\theta = 90^\circ$ and you will get the solution for a falling particle with air resistance.

PROBLEM 3 - Euler angles and a heavy spinning top

a. Euler angles are the natural generalized coordinates for a heavy spinning top as ϕ describes **precession** around the z-axis, θ marks **inclination** with respect to the vertical and ψ denotes **rotation** around the body axis z'.

The transformation of the ω-vector between fixed and rotating coordinate systems is achieved by decomposing the angular velocity vector in the components of Euler angles, where $\omega_\phi = \dot{\phi}$, $\omega_\theta = \dot{\theta}$ and $\omega_\psi = \dot{\psi}$.

$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi \tag{24}$$

The interrelationship between the components of $\vec{\omega}$ are the following:

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \tag{25}$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \tag{26}$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \tag{27}$$

This leads to kinetic energy $T = \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2$

The Lagrangian becomes

$$L = \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi} cos\theta)^2 - mgh\cos\theta \tag{28}$$

Lagrangian reveals immediately that ϕ and ψ are cyclic (do not appear explicitly), and therefore, the associated **canonical momenta** ($p_k = \partial L/\partial\dot{q}_k$) are conserved. Correspondingly, the angular velocity component $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$ that corresponds to the "spin" around the third principal body axis is the constant component of angular velocity.

b. By taking the partial derivatives of the Lagrange equation and applying the short-hand notation for ω_3, Lagrange equations become:

$$\frac{d}{dt}(I\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta) = 0 \tag{29}$$

$$I\ddot{\theta} - I\dot{\phi}^2\cos\theta\sin\theta + I_3\omega_3\dot{\phi}\sin\theta - mgh\sin\theta = 0 \tag{30}$$

$$\frac{d}{dt}(I_3\omega_3) = 0 \tag{31}$$

The above forms for ϕ and ψ are more illustrative than carrying out the time derivative explicitly, but both choices are valid, of course. Further, the explicit forms of p_ϕ and p_ψ can be seen inside the parentheses, respectively.

$$p_\phi = I\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta \tag{32}$$

$$p_\psi = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = I_3\omega_3 \tag{33}$$

These correspond to the angular momenta that involve precession and spinning of the top. Also, the fact that ω_3 is a constant becomes more clear through the latter equation.

Finally, the system is conservative meaning that **total energy** is conserved as well.

Note: One has to justify why there is no torque component N_3 if one wishes to use the Euler equation to show that ω_3 is a constant. Note that gravity causes some torque!

PROBLEM 4 - Relativistic collision

a. Let us start from the general properties that. According to the covariant 3+1 formulation, the 4-vector for an **event** is defined as

$$x_\mu = (x, y, z, ict) \tag{34}$$

In order to get **4-velocity**, one needs to take a derivative with respect to propert time τ

$$u_\mu = \frac{dx_\mu}{d\tau} = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, ic\frac{dt}{d\tau}\right) = \gamma(\mathbf{v}, ic) \tag{35}$$

Further, moving on to **4-momentum** is straigthforward

$$P_\mu = mu_\mu = \gamma(m\mathbf{v}, icm) = \gamma(\mathbf{p}, icm) = (\gamma\mathbf{p}, iE/c) \tag{36}$$

where we have used that total energy is $E = \gamma mc^2$. The product $P_\mu P_\mu$ is Lorentz invariant and most convenient to evaluate in the rest frame $P_\mu = (0, imc)$

$$P_\mu P_\mu = -m^2c^2 \tag{37}$$

This relation will appear very useful in the following.

b. We have conservation of 4-momentum

$$P_\mu^\pi + P_\mu^n = P_\mu^K + P_\mu^\Lambda \tag{38}$$

Let us arrange the terms such that

$$\begin{aligned}
P_\mu^\Lambda P_\mu^\Lambda &= (P_\mu^\pi + P_\mu^n - P_\mu^K)^2 \\
&= (\gamma_\pi \mathbf{p}_\pi + \gamma_n \mathbf{p}_n - \gamma_K \mathbf{p}_K)^2 - (E_\pi/m_\pi + E_n/m_n - E_K/m_K)^2 \\
&= (\gamma_\pi^2 \mathbf{p}_\pi^2 - E_\pi^2/m_\pi^2) + (\gamma_n^2 \mathbf{p}_n^2 - E_n^2/m_n^2) + (\gamma_K^2 \mathbf{p}_K^2 - E_K^2/m_K^2) \\
&\quad -2E_\pi E_n/(m_\pi m_n) + 2E_\pi E_K/(m_\pi m_K) + 2E_n E_K/(m_n m_K) \\
&\quad +2\gamma_\pi \gamma_n \mathbf{p}_\pi \cdot \mathbf{p}_n - 2\gamma_\pi \gamma_K \mathbf{p}_\pi \cdot \mathbf{p}_K - 2\gamma_n \gamma_K \mathbf{p}_n \cdot \mathbf{p}_K \\
&= \gamma_\Lambda^2 \mathbf{p}_\Lambda^2 - E_\Lambda^2/m_\Lambda^2 = -m_\Lambda^2 c^2
\end{aligned} \tag{39}$$

Let us note that neutron is at rest upon collision, thereby $\mathbf{p}_\pi \cdot \mathbf{p}_n = \mathbf{p}_n \cdot \mathbf{p}_K = 0$ and $E_n = m_n c^2$. Further, the angle between pion and kaon is 90 degrees, i.e. $\mathbf{p}_\pi \cdot \mathbf{p}_K = 0$.

For each particle: $P_\mu P_\mu = \gamma^2 \mathbf{p} - E^2/m^2 = -m^2 c^2$

This enables us to modify the equation above such that

$$-m_\Lambda^2 c^2 = -m_\pi^2 c^2 - m_n^2 c^2 - m_K^2 c^2 + 2m_n E_K/c^2 - 2(m_n c^2 - E_K)E_\pi/c^2 \quad (40)$$

$$\frac{E_\pi}{c^2} = \frac{m_\Lambda^2 c^2 - m_\pi^2 c^2 - m_n^2 c^2 - m_K^2 c^2 + 2m_n E_K}{2(m_n c^2 - E_K)} \quad (41)$$

The creation of kaon corresponds to a threshold $E_K = m_K c^2$

$$\frac{E_\pi}{c^2} \geq \frac{m_\Lambda^2 - m_\pi^2 - m_n^2 - m_K^2 + 2m_n m_K}{2(m_n - m_K)} \quad (42)$$

Finally, let us note that total energy is $E = T + mc^2$ and write for the kinetic energy of pion

$$\frac{T_\pi}{c^2} \geq \frac{m_\Lambda^2 - m_\pi^2 - m_n^2 - m_K^2 + 2m_n m_K}{2(m_n - m_K)} - m_\pi \quad (43)$$

In conclusion, this is the kinetic energy requirement for the incoming pion to create the aforementioned relativistic reaction.

PROBLEM 5 - Coriolis effect as seen from outside

a. Note that we are looking at the situation from an inertial frame (fixed stars).

Let us consider the trajectory of the falling pellet as a general two-body problem where Earth is considered as a point mass represented by its centre. The corresponding trajectory of the pellet is naturally a conical intersection, an ellipse (with eccentricity $\epsilon \sim 1$), and its location at the top of the tower (beginning, rest) corresponds to the apoapsis (i.e. the farthest point from the centre) with zero radial velocity ($\dot{r} = 0$). As the pellet drops, it naturally drops down to Earth's surface $r = R$ while Earth has rotated by a certain amount.

The horizontal velocity and angular momentum ℓ of the pellet are related to the rotation of Earth:

$$v_{hor} = r\omega \cos\lambda = (R + h)\omega \cos\lambda \quad (44)$$

$$\ell = mrv_{hor} = m(R + h)^2 \omega \cos\lambda \quad (45)$$

Here ω is the angular momentum of Earth, λ is the latitude and h is the height of the tower.

b. The solution of the Kepler problem reads now (p is **orbit parameter**!)

$$\frac{p}{r} = 1 - \epsilon \cos\theta \tag{46}$$

because we have chosen $\theta = 0$ in the beginning (apoapsis). Let us take the ratio of r and $R + h$ and modify the corresponding equation such that

$$r = \frac{(1-\epsilon)(R+h)}{1-\epsilon\cos\theta} \tag{47}$$

where θ corresponds to the orbit angle of any point on the elliptical orbit.

c. The constant areal velocity (Kepler II) is written as

$$\frac{1}{2}r^2\frac{d\theta}{dt} = \frac{\ell}{2m} \tag{48}$$

or $\ell = mr^2\dot{\theta}$ which is a constant and defined in **a**. Modifying the equation of differential quantities, substituting ℓ and r and integration on both sides leads to

$$t = \frac{m}{\ell}\int_0^\theta r^2 d\theta = \frac{m}{m(R+h)^2\omega\cos\lambda}\int_0^\theta \frac{(1-\epsilon)^2(R+h)^2}{(1-\epsilon\cos\theta)^2}d\theta \tag{49}$$

$$\implies \quad t = \frac{1}{\omega \cos\lambda} \int_0^{\theta} \frac{(1-\epsilon)^2}{(1-\epsilon\cos\theta)^2} d\theta \tag{50}$$

d. Let us select the hitting point $\theta = \theta_0 \implies r = R$ and use the same relation as above in **b**.

$$\begin{aligned} \frac{R+h}{R} &= 1 + \frac{h}{R} = \frac{1-\epsilon\cos\theta_0}{1-\epsilon} \\ &= \frac{1-\epsilon(1-2\sin^2(\theta_0/2))}{1-\epsilon} = 1 + \frac{2\epsilon}{1-\epsilon}\sin^2(\theta_0/2) \end{aligned} \tag{51}$$

$$\implies \quad \frac{h}{R} = \frac{2\epsilon}{1-\epsilon}\sin^2(\theta_0/2) \approx \frac{\epsilon\theta_0^2/2}{2(1-\epsilon)} \tag{52}$$

where we have used a simple Taylor expansion in the last step.

e. Similarly, expand the integral in **c** and apply the Taylor expansion:

$$\implies \quad t = \frac{1}{\omega\cos\lambda} \int_0^{\theta} \frac{d\theta}{[1 + 2\epsilon/(1-\epsilon)\sin^2(\theta/2)]^2} \tag{53}$$

$$\implies \quad t \approx \frac{1}{\omega \cos\lambda} \int_0^{\theta} \frac{d\theta}{[1 + \epsilon\theta^2/(2(1-\epsilon))]^2} \tag{54}$$

Substitute the result from **d** for $\epsilon/2(1-\epsilon)$ and denote the time for reaching the landing point as $t(\theta_0) = T$

$$\implies \quad T \approx \frac{1}{\omega \cos\lambda} \int_0^{\theta_0} \frac{d\theta}{[1 + (h\theta^2/R\theta_0^2)]^2} \tag{55}$$

This was the result that one was suppose to derive here. Further, as a bonus, by taking another Taylor expansion and integrating, we get

$$\implies \quad T \approx \frac{1}{\omega \cos\lambda} \int_0^{\theta_0} \left(1 - \frac{2h}{R\theta_0^2}\theta^2\right) d\theta = \frac{1}{\omega \cos\lambda} \left(1 - \frac{2h}{3R}\right)\theta_0 \tag{56}$$

f. Based on the latest result, it is straightforward to convince ourselves that

$$\theta_0 \approx \frac{\omega T \cos\lambda}{1 - 2h/3R} \approx \omega T \cos\lambda \left(1 + \frac{2h}{3R}\right) \tag{57}$$

where we have taken a Taylor approximation once more. This the starting point of the last step in our journey. Let us not forget that **Earth rotates** an angle ωT toward East while the pellet is travelling on its elliptical orbit. A point on Earth's surface moves thereby $R\omega \cos\lambda$. On the other hand, the particle is being deflected toward East by an amount $R\theta_0$ on its orbit. Thus, the net devitiation becomes

$$d = R\theta_0 - R\omega\cos\lambda = \frac{2}{3}h\omega T\cos\lambda \tag{58}$$

As a final step, we can approximate the falling time based on the standard description of free fall in a non-rotating frame of reference $T \approx \sqrt{2h/g}$. Substititute this for deflection, and voilà, we shall reach the ultimate goal!

$$d \approx \frac{1}{3}h\omega\cos\lambda\sqrt{\frac{8h^3}{g}} \tag{59}$$

The deflection is toward East and the result is exactly the same as derived previously via applying the equation for Coriolis force. Note that we have made some approximations with both methods.