## PROBLEM 1. Rotation of Earth

**A.** The discussion could be started by asking if the person making such silly claims has ever jumped on a conveyor belt? This is of course for linear motion, and we extend it here to rotations. Felix had the same radial velocity as the ground while he started his journey upwards. In principle, this velocity changes (centripetal motion) as a result of conservation of *his* angular momentum, while the distance from the origin (Earth's center) changes upon ascend. However, the order of magnitude of $h$ in comparison to $R$ is so small that the effect becomes negligible. On the other hand, the Coriolis effect is dependent on velocity, and it is very small for this part of the journey. One can also take into account that the radial velocity of atmosphere is, on average, the same as that of the ground.

**B.** Felix starts the free fall from 39 km (checkpoint 1) and reaches the maximum velocity at checkpoint 2 (no air resistance). Let us first calculate how much time it takes and what is the altitude of checkpoint 2. It is elementary to calculate that $t = v_{max}/g = 38.45$ s and $\Delta h = v_{max}^2/(2g) = 7253$ m, which leads to altitude of 31.7 km.

Next, we calculate the acceleration in the rotational frame $\mathbf{a}_s = \mathbf{a}_r - 2\omega \times \mathbf{v}_r$. Assume that lateral velocities $\dot{x} \approx 0$ and $\dot{y} \approx 0$, while $\dot{z} = -gt$. The angular velocity of Earth is $(-\omega \cos\alpha, 0, \omega \sin\alpha)$ in the local coordinate system (see the figure in the problem set). Let us write the cross-product $\omega \times \mathbf{v}_r$ as a determinant and it leads to

$$\omega \times \mathbf{v}_r = -\omega g t \cos\alpha \mathbf{e}_j \tag{1}$$

Going back to the relation between accelerations, the acceleration components become $(a_r)_x = \ddot{x} = 0$, $(a_r)_y = \ddot{y} = 2\omega g t \cos\alpha$, and $(a_r)_z = \ddot{z} = -gt$. Integrate the middle one twice by assuming $y(0) = \dot{y}(0) = 0$ and take into account that $t = \sqrt{2\Delta h/g}$. This leads to our final outcome

$$y(t) = d = \frac{1}{3}\omega \cos\alpha \sqrt{8\Delta h^3/g} \tag{2}$$

Insert the values of initial conditions and $\Delta h$, and the result becomes 11.5 m deflection in East. In total, one can deduce that the deflection caused by the Coriolis effect is well below 100 m for the whole experiment.

## PROBLEM 2. True of false

**i.** TRUE. We can use the method of Lagrange's undetermined multipliers for solving non-holonomic constraints with velocities (semi-holonomic constraints) and cases where there is an inequality in the constraint equation that would otherwise be holonomic / semi-holonomic.

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = -\sum_{l=1}^{m} \lambda_l \frac{\partial f_l}{\partial \dot{q}_k} = -\sum_{l=1}^{m} \lambda_l a_{l,k} = Q_k, \qquad k = 1, 2, ..., n \tag{3}$$

The nice feature is that this equation can also be applied for holonomic constraints in case we want to know the forces of constraints..

**ii.** FALSE. The Lagrangian function can be transformed to another Lagragian via the transformation

$$L' = L + \frac{dF(q,t)}{dt} \tag{4}$$

In the same context, remember also that the potential energy zero-level is arbitrary.

**iii.** FALSE. In general, a bound orbit is not the same as a closed orbit. Also other central potentials result in bound orbits but they are not closed.

**iv.** FALSE. Hamilton's characteristic function $W$ applies for cases where the Hamiltonian itself does not depend on time. We must solve the Hamilton-Jacobi equation some other way. (For those interested, it goes via an ansatz function $S(x,t) = f(t)x + g(t)$. We did not learn this technique in this course.)

**v.** FALSE. Starting from the classical mechanics point of view, diatomic molecules do not have transversal vibrational modes which are necessary for IR activity (moving center of charge) and absorption of light (heat). $N_2$ and $O_2$ are far more abundant than $CO_2$ in the atmosphere. It is true that the $CO_2$ concentration exceeds greatly those of larger molecules such as $CH_4$ which are even worse in their ability to absorb light. Actually, the most abundant green house gas (GHG) is $H_2O$. The $H_2O$ and $CO_2$ levels are coupled, and thereby emissions of the latter affect also the former (clouds). One can of course argue about the definition of GHG in the context of water.

## PROBLEM 3. Particle in cylinder cavity with water

**a** Let us start by identifying the analogy with the problem of simple pendulum. The kinetic energy and potential energy will have similar expressions in the two cases. For treating this problem, the obvious choice is polar coordinates, where $R$ (radius) is fixed. Thereby, a convenient generalized coordinate of treating this problem is the angle $\theta$, measured from the bottom of cavity. The corresponding Lagrangian becomes

$$L = \frac{1}{2} m R^2 \dot{\theta}^2 - mgR(1 - \cos\theta) \tag{5}$$

There is a dissipative force due to liquid with respect to the radial velocity $v = R\dot{\theta}$ which corresponds to the generalized force on the other size of the Lagrange equation (note the sign)

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = kmR\dot{\theta} \tag{6}$$

After taking the derivatives and dividing by common factors, the equation of motion becomes

$$\ddot{\theta} + \frac{k}{R}\dot{\theta} + \frac{g}{R}\sin\theta = 0 \tag{7}$$

**b** Let us now introduce the new (auxiliary) constants $\delta = k/(2R)$ and $\omega_0 = \sqrt{g/r}$. We also assume small oscillations where $\sin\theta \sim \theta$. The differential equation becomes

$$\ddot{\theta} + 2\delta\dot{\theta} + \omega_0^2\theta = 0 \tag{8}$$

We observe that the equation above is a second order homogeneous differential equation. The solution of this achieved via the roots of the characteristic equation

$$r^2 + 2\delta r + \omega_0^2 = 0 \tag{9}$$

where $r = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$. Further, let us denote $\alpha = \sqrt{\delta^2 - \omega_0^2}$.
We have now three scenarios, where the first one corresponds to $\delta^2 > \omega_0^2$. This leads to

$$\theta = Ae^{r_1 t} + Be^{r_2 t} = e^{-\delta t}\left(C_1 e^{\alpha t} + C_2 e^{-\alpha t}\right) \tag{10}$$

where $A$, $B$, $C_1$ and $C_2$ are undetermined constants. This form of solution clearly describes a decaying overdamped case of oscillation.

Similarly, $\delta^2 = \omega_0^2$ leads to critical damping $\theta = e^{-\delta t}(C_1 + C_2 t)$.

For $\delta_2 < \omega_0^2$ we will introduce another parameter $\omega = \sqrt{\omega_0^2 - \delta^2}$ and the solution of the differential equation becomes

$$\theta = e^{-\delta t}\left(C_1 e^{i\omega t} + C_2 e^{-i\omega t}\right) \tag{11}$$

The first part of the equation is a decaying function, while the parentheses describe oscillation. This is the underdamped case of oscillation.

**Note:** The solution of this problem essentially the same as for the damped simple pendulum.

**PROBLEM 4. Inertia tensor of a cube**

**a.** The cube is fixed in the middle of one of its edges while the sides are aligned with the cartesian coordinate axes. The integration limits are therefore $[0, a]$, $[0, a]$ and $[-a/2, a/2]$ for $x_1$, $x_2$ and $x_3$, respectively. The inertia tensor is defined as

$$I_{ij} = \int_V \rho(\mathbf{r})\Big(\delta_{ij} r^2 - r_i r_j\Big) dV \qquad (12)$$

The density is a constant and we can write out the first tensor component

$$\begin{aligned} I_{11} &= \rho \int_V \Big(r^2 - x_1^2\Big) dV = \rho \int_{-a/2}^{a/2} dx_3 \int_0^a (x_2^2 + x_3^2) dx_2 \int_0^a dx_1 & (13) \\ &= \ldots = \frac{5}{12}\rho a^5 = \frac{5}{12} M a^2 & (14) \end{aligned}$$

by taking into account that $M = \rho a^3$. Let us remember that the inertia tensor is Hermitian. This results in that the off-diagonal component are $I_{21} = I_{12}$, $I_{31} = I_{13}$ and $I_{32} = I_{23}$.

The remaining components to calculate are

$$\begin{aligned}
I_{12} &= -\rho \int_0^a x_1 dx_1 \int_0^a x_2 dx_2 \int_{-a/2}^{a/2} dx_3 = ... = -\frac{1}{4} Ma^2 &(15)\\
I_{13} &= -\rho \int_0^a x_1 dx_1 \int_{-a/2}^{a/2} x_3 dx_3 \int_0^a dx_2 = ... = 0 &(16)\\
I_{22} &= \rho \int_{-a/2}^{a/2} dx_3 \int_0^a (x_1^2 + x_3^2) dx_1 \int_0^a dx_2 = ... = \frac{5}{12} Ma^2 &(17)\\
I_{23} &= -\rho \int_0^a x_2 dx_2 \int_{-a/2}^{a/2} x_3 dx_3 \int_0^a dx_1 = ... = 0 &(18)\\
I_{33} &= \rho a \int_0^a dx_1 \int_0^a (x_1^2 + x_2^2) dx_2 = ... = \frac{2}{3} Ma^2 &(19)
\end{aligned}$$

The corresponding final matrix can now be written by using the short hand notation $b = Ma^2$.

**b.** The determinant of the inertia tensor is simple to solve and results in a polynomial

$$\left(\frac{2}{3}b - I\right)\left[\left(\frac{5}{12}b - I\right)\left(\frac{5}{12}b - I\right) - \left(\frac{1}{4}b\right)^2\right] = 0 \tag{20}$$

We see immediately that one of the roots is $I = \frac{2}{3}b$. What is left is the polynomial in square-brackets, which becomes

$$I^2 - \frac{5}{6}bI + \frac{1}{9}b^2 = 0 \tag{21}$$

The remaining roots can be calculated from the general solution of a second order polynomial, and they become $I = \frac{2}{3}b$ and $I = \frac{1}{6}b$. Obviously, we have two degenerate eigenvalues.

The task did not include computing the eigenvectors, *i.e.* principal axesof inertia, but let us give them here as well: [-1,1,0], [1,1,0] and [0,0,1]. One can add a constant $1/\sqrt{2}$ for the two former ones for normalization.

**PROBLEM 5. Meson decay**

The task can be solved by considering the conservation of energy and linear momentum. For energy conservation, we can use the relativistic dispersion relation $E^2 = m^2c^4 + p^2c^2$ and the fact that linear momentum is conserved, that is $p_\mu = -p_\nu$. We can denote the magnitude of each linear momentum simply as $p$. The equation for energy conservation $E_\pi = E_\mu + E_\nu$ becomes

$$m_\pi c^2 = \sqrt{m_\mu^2 c^4 + p^2c^2} + pc, \tag{22}$$

where we have the initial meson on the lefthand side and products on the righthand side. Note that the initial meson has only rest mass, and neutron has no mass, $E_\nu = pc$. Let us next solve $p$ in this equation. After some steps

$$p = \frac{m_\pi^2 - m_\mu^2}{2m\pi}c. \tag{23}$$

Insert this in the total energy of $\mu$-meson

$$E_\mu = \sqrt{m_\mu^2 c^4 + p^2c^2} = ... = \frac{c^2}{2m_\pi}(m_\pi^2 + m_\mu^2). \tag{24}$$

Kinetic energy of $\mu$-meson is simply the difference between total and rest energies

$$T_\mu = E_\mu - m_\mu c^2 = \frac{c^2}{2m_\pi}(m_\pi^2 + m_\mu^2) - m_\mu c^2 = \ldots = \frac{(m_\pi - m_\mu)^2}{2m_\pi} c^2 \tag{25}$$