TFY4345 Classical Mechanics Suggested Solution to Exam November 27, 2023

Part I

1.1 B Such a constraint, expressed in terms of an equation, is called holonomic.

1.2 B In the element ε_{321} of the Levi-Civita tensor, the indices come in a 'non-cyclic' order, but they are all different. Then the value is -1.

1.3 A An angle is dimensionless. Then, $p = \partial L/\partial\dot{\theta}$ has the unit Js, i.e., the unit of an angular momentum.

1.4 E A coordinate not present in the Lagrangian is called cyclic.

1.5 E Here, the kinetic energy $T = m(\dot{x}^2 + \dot{y}^2)/2$ will not be a function of θ. Then p_θ is conserved.

1.6 D $n = 3N - k = 3 \cdot 3 - 3 = 6$ independent coordinates.

1.7 B We have 4 independent coordinates: x for the sliding support at A and an angle θ_j $(j = 1, 2, 3)$ between (e.g.) the vertical axis and each massless rod.

1.8 C $T = (m_1 + m_2)\dot{x}^2/2$ and $V = -m_1 gx - m_2 g(\ell - x)$ gives

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx + m_2 g\ell$$

1.9 F $p_2 = \partial L/\partial\dot{x}_2 = m\dot{x}_2 + qA_2$.

1.10 B $B_i = \varepsilon_{ijk}\partial_j A_k$. Here, $B_1 = B_2 = 0$ and $B_3 = \varepsilon_{312}\partial_1 A_2 + \varepsilon_{321}\partial_2 A_1 = B_0 - (-B_0) = 2B_0$.

1.11 E $\boldsymbol{E} = -\partial\boldsymbol{A}/\partial t - \nabla\phi = -E_0(\hat{x} + \hat{y} + \hat{z})$.

1.12 C $p_1 = \partial L/\partial\dot{q}_1 = c_3 q_1$.

1.13 C $r_{\text{min}}/r_{\text{max}} = [p/(1+\varepsilon)]/[p/(1-\varepsilon)] = (1-\varepsilon)/(1+\varepsilon) = 0.79/1.21 = 0.65$.

1.14 A Compared with the 'usual' example of rotation around the x_3 axis, the 3 axis now plays the role of the 1 axis, and the 1 axis now plays the role of the 2 axis. Therefore, the rotation matrix is

$$\begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$$

1.15 A $v = c \cdot (5/6 + 7/8)/(1 + 5 \cdot 7/6 \cdot 8) = 82c/83$.

1.16 D In this question, and the next, the angular velocity of Earth is $\omega = 2\pi/T = [2\pi/(24 \cdot 3600)]\ \text{s}^{-1}$. Then $a_{\text{Cor}} = 2v_b\omega = 2 \cdot 15 \cdot 2\pi/(24 \cdot 3600)\ \text{m/s}^2 = 2.2\ \text{mm/s}^2$.

1.17 D $a_{\text{Cen}} = \omega^2 r = (2\pi/(24 \cdot 3600))^2 \cdot 6.378 \cdot 10^6\ \text{m/s}^2 = 34\ \text{mm/s}^2$.

1.18 F We have $E^2 = (pc)^2 + (mc^2)^2$, i.e., $m = \sqrt{E^2 - (pc)^2}/c^2 = \sqrt{500^2 - 400^2}$ MeV$/c^2$ = 300 MeV$/c^2$.

1.19 F $p = \partial F_1/\partial q = Q - 2q$, i.e., $Q(q,p) = p + 2q$. Then, $P = -\partial F_1/\partial Q = -q + 2Q = -q + 2\cdot(p+2q) = 3q + 2p$.

1.20 C A key ingredient in Unni's Cinnamon Cake is Cinnamon.

Part II

2

a) Stretch of left spring:

$$(x_2 - x_1) - (x_{20} - x_{10}) = \eta_2 - \eta_1$$

Stretch of right spring:

$$(x_3 - x_2) - (x_{30} - x_{20}) = \eta_3 - \eta_2$$

(Compression when $\eta_2 - \eta_1$ or $\eta_3 - \eta_2$ negative.) Potential energy:

$$\begin{aligned} V &= \frac{1}{2}k(\eta_2 - \eta_1)^2 + \frac{1}{2}k(\eta_3 - \eta_2)^2 \\ &= \frac{1}{2}k(\eta_1^2 + 2\eta_2^2 + \eta_3^2 - \eta_1\eta_2 - \eta_2\eta_1 - \eta_2\eta_3 - \eta_3\eta_2) \end{aligned}$$

The matrix $\boldsymbol{V}$ is:

$$\boldsymbol{V} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

Kinetic energy:

$$T = \frac{1}{2}m_i\dot{x}_i^2 = \frac{1}{2}m_i\dot{\eta}_i^2$$

The matrix $\boldsymbol{T}$ is:

$$\boldsymbol{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix}$$

b) The eigenfrequencies are given by the secular equation

$$\begin{vmatrix} k - \omega^2 m & -k & 0 \\ -k & 2k - 2\omega^2 m & -k \\ 0 & -k & k - 4\omega^2 m \end{vmatrix} = 0$$

If we extract a common factor k^3 and introduce the dimensionless factor $\alpha = m\omega^2/k$, the resulting equation is

$$\begin{vmatrix} 1-\alpha & -1 & 0 \\ -1 & 2-2\alpha & -1 \\ 0 & -1 & 1-4\alpha \end{vmatrix} = 0$$

We expand the determinant by (e.g.) the upper row and find

$$(1-\alpha)(2-2\alpha)(1-4\alpha) - (1-\alpha) - (1-4\alpha) = 0$$

or

$$8\alpha^3 - 18\alpha^2 + 7\alpha = 0.$$

The normal mode with $\alpha = 0$ corresponds to pure translation of the system along the x axis. This is of no interest here. The two non-zero solutions are

$$\alpha = \frac{18 \pm \sqrt{18^2 - 4 \cdot 8 \cdot 7}}{2 \cdot 8} = \frac{18 \pm 10}{16} = \frac{9 \pm 5}{8},$$

i.e.,

$$\alpha_1 = \frac{1}{2} \quad , \quad \alpha_2 = \frac{7}{4}.$$

This corresponds to the frequencies

$$f_1 = \frac{\omega_1}{2\pi} = \frac{\sqrt{k\alpha_1/m}}{2\pi} = \frac{\sqrt{k/2m}}{2\pi} \simeq 6.2 \text{ Hz}$$

and

$$f_2 = \frac{\omega_2}{2\pi} = \frac{\sqrt{k\alpha_2/m}}{2\pi} = \frac{\sqrt{7k/4m}}{2\pi} \simeq 11.5 \text{ Hz}.$$

c) We insert the smallest value $\alpha = 1/2$ into the equation

$$\begin{pmatrix} 1-\alpha & -1 & 0 \\ -1 & 2-2\alpha & -1 \\ 0 & -1 & 1-4\alpha \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

and find

$$\begin{aligned} \frac{1}{2}A_1 - A_2 &= 0 \\ -A_1 + A_2 - A_3 &= 0 \\ -A_2 - A_3 &= 0 \end{aligned}$$

Here, we have only two independent equations, so we can only find e.g. A_2 and A_3 expressed in terms of A_1. If we set $A_1 = 1$, we find $A_2 = 1/2$ and $A_3 = -1/2$. In this normal mode, m and $2m$ oscillate in phase, and out of phase with $4m$, with amplitudes that ensures a mass center at rest.

3

a) Potential energy: $V = -mg\ell\cos\theta$.
Kinetic energy (upper mass 1 and lower mass 2):

$$T_1 = \frac{1}{2}m\dot{x}^2 \quad , \quad T_2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2).$$

For the two masses, $x = \pm(\ell/2)\sin\theta$, and for the lower mass, $y = -\ell\cos\theta$. Then, $\dot{x} = \pm(\ell\dot{\theta}/2)\cos\theta$ and $\dot{y} = \ell\dot{\theta}\sin\theta$. We insert this into T_1 and T_2 and find, after using the hint,

$$L = T - V = T_1 + T_2 - V = \frac{m\ell^2\dot{\theta}^2}{4}\left(1 + \sin^2\theta\right) + mg\ell\cos\theta.$$

b) For the equation of motion, we need these derivatives:

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= \frac{m\ell^2\dot{\theta}^2}{2}\sin\theta\cos\theta - mg\ell\sin\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= \frac{m\ell^2\dot{\theta}}{2}\left(1 + \sin^2\theta\right) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= \frac{m\ell^2\ddot{\theta}}{2}\left(1 + \sin^2\theta\right) + m\ell^2\dot{\theta}^2\sin\theta\cos\theta \end{aligned}$$

We multiply the Lagrange equation with $2/m\ell^2$ and find

$$\ddot{\theta}(1+\sin^2\theta) + \sin\theta\left(\frac{2g}{\ell} + \dot{\theta}^2\cos\theta\right) = 0.$$

c) To linear order, $\sin\theta = \theta$ and $\cos\theta = 1$. Then,

$$\ddot{\theta} + \frac{2g}{\ell}\theta = 0.$$

This is a harmonic oscillator with frequency $\omega = \sqrt{2g/\ell}$.

d) We have already

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = \frac{m\ell^2\dot{\theta}}{2}\left(1+\sin^2\theta\right)$$

which means that

$$\dot{\theta} = \frac{2p_\theta}{m\ell^2(1+\sin^2\theta)}.$$

We insert this expression for $\dot{\theta}$ in L and find

$$H = p_\theta\dot{\theta} - L = \frac{p_\theta^2}{m\ell^2(1+\sin^2\theta)} - mg\ell\cos\theta.$$

4 When the impact parameter is $s = a/\sqrt{2}$, the point particle collides with the surface of the disc with an angle $\pi/4$ between the incoming direction and the surface normal. The collision is specular, so the scattering angle is $\pi/2$.