Institutt for fysikk, NTNU
TFY4155/FY1003 Elektrisitet og magnetisme
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Solution to øving 11

Exercise 1

a) This system can be viewed as three resistances connected in series: the two 60 cm long Cu wires and the resistance $R = 20\ \Omega$. The resistance of the two Cu wires becomes

$$R_A = \frac{l}{\sigma A} = \cdot \frac{1.20\text{m}}{5.8 \cdot 10^7\ \Omega^{-1}\text{m}^{-1} \cdot 2 \cdot 10^{-6}\text{m}^2} = 0.01\ \Omega$$

The same current I passes through the whole system. It is

$$I = \frac{V}{R + R_C} = \frac{1.5\ \text{V}}{20.01\ \Omega} = 0.07496\ \text{A} \simeq 0.075\ \text{A}$$

according to Ohm's law. Thus, we obtain the voltage drops

$$V_R = RI = 20\ \Omega \cdot 0.075\ \text{A} \simeq 1.5\ \text{V}$$

over the resistance R and

$$V_C = R_C I = 0.01\ \Omega \cdot 0.075\ \text{A} = 0.00075\ \text{V}$$

over the two Cu wires together. In conclusion: Negligible voltage drop in the two Cu wires.

b) We found the current I in *a*) above. The dissipated effect in the resistance R becomes

$$P = V_R I = 1.5\ \text{V} \cdot 0.075\ \text{A} = 0.1125\ \text{W} \simeq 0.11\ \text{W}$$

c) Here, we must first find the density of free electrons n. Next, we may use $I = j \cdot A = nevA$ in order to calculate the mean drift velocity v.
In Cu, we have a mass density 8960 kg pr m^3. This corresponds to 8960/0.06354 mol = 141014 mol = 141014 $\cdot 6.02 \cdot 10^{23}$ atomer = $8.49 \cdot 10^{28}$ atoms, and hence equally many free electrons, assuming one free electron pr Cu atom. Mean drift velocity becomes

$$v = \frac{I}{neA} = \frac{0.075}{8.49 \cdot 10^{28} \cdot 1.6 \cdot 10^{-19} \cdot 2 \cdot 10^{-6}} = 2.76 \cdot 10^{-6}\ \text{m/s} = 2.76\mu\text{m/s}$$

Average thermal velocity for the electrons may be estimated by setting the kinetic energy equal to the thermal energy:

$$\begin{aligned} \frac{1}{2}mv^2 &= \frac{3}{2}k_B T \\ \Rightarrow v &= \sqrt{\frac{3k_B T}{m}} \simeq 10^5\ \text{m/s} \end{aligned}$$

Here, $k_B = 1.38 \cdot 10^{-23}$ J/K is Boltzmann's constant. We see that the mean drift velocity is roughly 11 orders of magnitude smaller than the average thermal velocity. In other words, it takes several hours for a given electron to get from one end to the other in our system!

Exercise 2

a) At first, we should try to realize that what we have here is the following circuit: [a parallel connection of R_1, R_2 and R_3] coupled in series with [a parallel connection of R_4 and $R_0 = 0$] in series with [R_5]. In other words, the resistance R_4 is "cut short", so that no current passes through R_4. (Alternatively: We have the same value for the potential on each side of R_4. Then, no current can pass through it.) Thus, the total resistance is

$$R = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} + R_5$$

b) It should be clear that the total current I in the circuit must be the same as the current I_5 passing through R_5. Further, it should also be clear that I must distribute itself on the three currents passing through R_1, R_2 and R_3: $I = I_1 + I_2 + I_3$. In *a*) above, we have already concluded that no current passes through R_4: $I_4 = 0$.
The voltage drop across the three upper resistances is the same:

$$V' = R_1 I_1 = R_2 I_2 = R_3 I_3$$

The voltage drop across R_5 is

$$V'' = R_5 I_5 = R_5 I = R_5 \frac{\mathcal{E}}{R}$$

These two together must equal the value of the voltage source:

$$\mathcal{E} = V' + V''$$

Thus

$$V' = \mathcal{E} - V'' = \mathcal{E} - R_5 \frac{\mathcal{E}}{R} = \mathcal{E}\left(1 - \frac{R_5}{R}\right)$$

And finally,

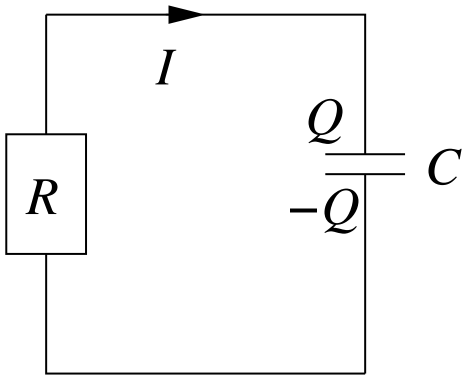
$$I_1 = \frac{V'}{R_1}$$

$$I_2 = \frac{V'}{R_2}$$

$$I_3 = \frac{V'}{R_3}$$

Exercise 3

The capacitor will be discharged by electrons moving from the negatively charged side, through the resistor, to the postively charged side. In other words, a positive current will run through R, from the positively charged plate.



In the figure, I has been drawn with direction towards the positive plate, despite the conclusion above, that positive I runs in the opposite direction. This is done, since I prefer to keep the relation

$$I = +\frac{dQ}{dt}$$

(and not with a minus sign). Kirchhoff's voltage rule yields

$$-RI - \frac{Q}{C} = 0$$

i.e.

$$\frac{dQ}{Q} = -\frac{dt}{RC}$$

Integration on both sides gives

$$Q = ke^{-t/RC}$$

and the initial condition $Q(0) = Q_0$ determines the integration constant k:

$$Q_0 = ke^0 = k$$

so that

$$Q(t) = Q_0 e^{-t/RC}$$

The current then becomes

$$I(t) = \frac{dQ}{dt} = -\frac{Q_0}{RC} e^{-t/RC}$$

As expected with a minus sign, so that positive current I runs counterclockwise in the figure above.

Exercise 4

a) Kirchhoff's voltage rule (K2) gives

$$\mathcal{E} = \frac{Q}{C}$$

Also, we have

$$I = \frac{dQ}{dt}$$

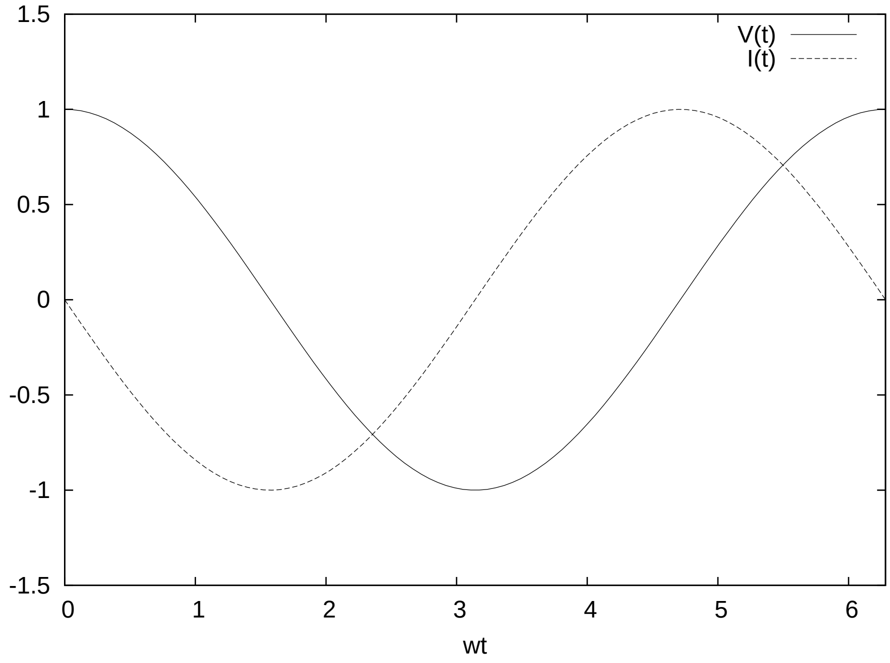
Hence:

$$Q(t) = V_0 C \cos \omega t$$

and

$$I(t) = \dot{Q} = -V_0 \omega C \sin \omega t$$

Sketch of $\mathcal{E}(t)$ (in the figure, $V(t)$, solid line) and $I(t)$ between $t = 0$ and $t = T = 2\pi/\omega$:



We see that the applied voltage $\mathcal{E}(t)$ and the resulting current $I(t)$ in the circuit are not at maximum, zero etc at the same instants. We say that we have a *phase difference* between $\mathcal{E}(t)$ and $I(t)$. For this simple circuit, with only a single capacitor, we see that I is zero when $\mathcal{E}$ is at a maximum value, and the other way around. Hence, the two have a phase difference of $\pi/2$. This is most clearly seen if we write I on the form

$$I(t) = I_0 \cos(\omega t - \alpha)$$

We have

$$-\sin \omega t = \cos(\omega t + \pi/2)$$

Hence, we see that $I_0 = V_0\omega C$ and $\alpha = -\pi/2$. The impedance of this simple circuit, i.e., a single capacitor with capacitance C, becomes

$$Z_C(\omega) = \frac{V_0}{I_0} = \frac{1}{\omega C}$$

Notice that the impedance increases when the angular frequency becomes smaller. In the limit $\omega \to 0$ the impedance Z_C approaches infinity, which is quite reasonable: No direct current through a (ideal) capacitor. In the opposite limit, $\omega \to \infty$, Z_C approaches zero, which is also reasonable: The charge on a given plate oscillates between $+V_0C$ and $-V_0C$, no matter what the frequency is. When the frequency is increased, the current in the circuit will increase linearly with the frequency. (Later, we will see how the magnetic field stops the current from increasing beyond limits.)

b) Kirchhoff's voltage rule (K2) gives

$$\mathcal{E} = \frac{Q}{C} = RI_R$$

whereas Kirchhoff's current rule (K1) gives

$$I = I_C + I_R$$

Also, we have

$$I_C = \frac{dQ}{dt}$$

Hence:

$$\begin{aligned} I_R(t) &= \frac{V_0}{R}\cos\omega t \\ Q(t) &= V_0C\,\cos\omega t \\ I_C(t) &= -\omega C V_0\,\sin\omega t = \omega C V_0\,\cos(\omega t + \pi/2) \end{aligned}$$

Total current delivered by the voltage source becomes

$$I(t) = \frac{V_0}{R}\cos\omega t - \omega C V_0\,\sin\omega t$$

We want to have $I(t)$ on the form

$$I(t) = I_0\,\cos(\omega t - \alpha)$$

with amplitude $I_0 = V_0/Z$, where Z is the impedance of R and C coupled in parallel, while α becomes the phase difference between $\mathcal{E}(t)$ and $I(t)$. We have

$$\cos(\omega t - \alpha) = \cos\omega t\,\cos\alpha + \sin\omega t\,\sin\alpha$$

Hence, upon direct comparison:

$$\begin{aligned} \frac{\cos\alpha}{Z} &= \frac{1}{R} \\ \frac{\sin\alpha}{Z} &= -\omega C \end{aligned}$$

These two equations, with the two unknowns Z and α, are easily solved, and we find

$$\begin{aligned} Z &= \frac{R}{\sqrt{1+(\omega RC)^2}} \\ I_0 &= \frac{V_0}{R}\sqrt{1+(\omega RC)^2} \\ \alpha &= -\arctan(\omega RC) \end{aligned}$$

In the limit $\omega \to 0$, we should recover well known results from the DC examples in the lectures, and indeed, we do: $Z \to R$ and $\alpha \to 0$ so that $I_0 \to V_0/R$. All the current goes through the resistor R, and the capacitor C now represents an open circuit where no direct current can run.

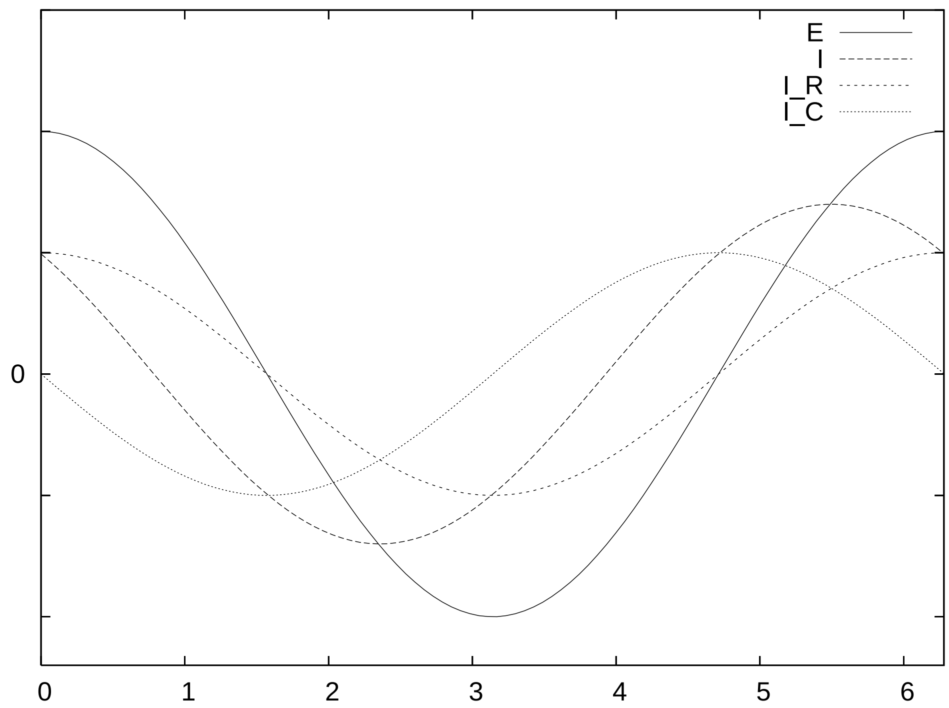
With the given numerical values we have

$$\omega RC = 2\pi \cdot 10^6 \cdot 10 \cdot 16 \cdot 10^{-9} = 1.0$$

so that

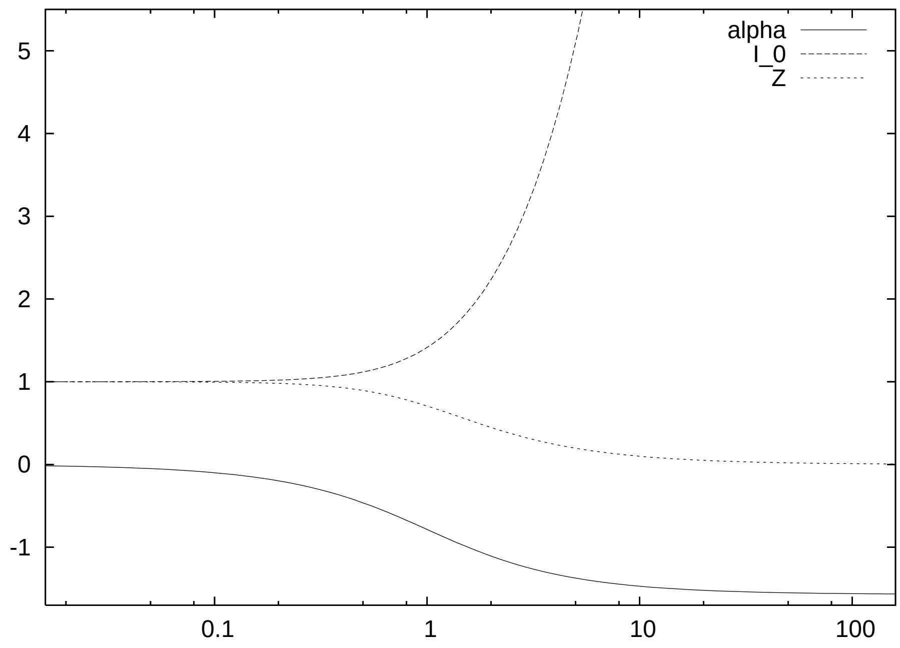
$$\begin{aligned} I_0 &= \frac{1.0}{10} \cdot \sqrt{2} = 0.14 \text{ A} \\ \alpha &= -\arctan 1.0 = -45^\circ \end{aligned}$$

Sketch of $\mathcal{E}(t)$, $I(t)$, $I_R(t)$ and $I_C(t)$:



Notice that I_R oscillates in phase with $\mathcal{E}$, whereas I_C has a phase difference of $\pi/2$ with respect to $\mathcal{E}$. The phase difference between $\mathcal{E}$ and the total current $I = I_R + I_C$ becomes, with the given numerical values, somewhere in between, i.e., 45 degrees.

Sketch of α, I_0 and Z (with α in radians and ωRC between 0.016 and 160 along the horizontal axis):



Notice that the phase difference α between the applied voltage $\mathcal{E}$ and the total current I is approximately zero for small frequencies. In that case, most of the current goes through the resistor ($I \simeq I_R$), and the circuit behaves roughly as it would have done without the capacitor present. For high frequencies, α is approximately -90 degrees. In this case, most of the current goes in the "branch" with the capacitor ($I \simeq I_C$), and the circuit behaves roughly as it would have done without the resistor present.

Hence, for low frequencies, the current amplitude I_0 is appoximately constant, i.e., independent of the frequency, and for high frequencies, I_0 increases linearly with the frequency.

We see that the transition between "low" and "high" frequencies happens where $\omega \simeq 1/RC = 1/\tau$, where τ is the time constant for such an RC circuit.

Exercise 5

1. The full region between $r = a$ and $r = b$ may be viewed as many resistances dR connected in series, where each resistor is a thin spherical shell with radius r and thickness dr:

$$dR = \frac{\rho\, dr}{4\pi r^2}$$

The total resistance is found by summing up all these individual resistances, i.e., by integrating from $r = a$ to $r = b$:

$$\begin{aligned} R &= \int dR \\ &= \int_a^b \frac{\rho\, dr}{4\pi r^2} \\ &= \frac{\rho}{4\pi}\Big|_a^b \left(-\frac{1}{r}\right) \\ &= \frac{\rho}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right) \end{aligned}$$

2. The given expression for the current I shows that we may here use Gauss' law for the electric field to determine I:

$$I = \frac{1}{\rho} \cdot \frac{Q}{\varepsilon_0}$$

We must assume that the charge entering into the inner conductor immediately distributes itself over the spherical surface at $r = a$ before is starts on its way through the material between $r = a$ and $r = b$.
The potential difference between the inner and outer conducting shell is easily determined, since we know the electric field $\boldsymbol{E}$:

$$\begin{aligned} \Delta V &= V_a - V_b \\ &= -\int_b^a E(r)\, dr \\ &= \frac{Q}{4\pi\varepsilon_0}\Big|_b^a \frac{1}{r} \\ &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \end{aligned}$$

From these expressions, it follows that the resistance is

$$R = \frac{\Delta V}{I} = \frac{\rho}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right)$$