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What is the frequency of an electron wave?

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Abstract

Particle–wave duality is a central tenet of quantum physics, and an electron has wave-like properties. Introductory texts discuss the wavelength–momentum relationship $\lambda = h/p$, but do not discuss the frequency–energy relationship. This is curious since a wave is periodic both in space and time. The discussion in more advanced texts is not satisfactory either since two different expressions for the frequency are given based on the relativistic and non-relativistic expression for the electron energy. The relativistic expression yields the correct frequency, and we explain why the expression based on the Schrödinger equation gives the *incorrect* expression. We argue that the electron frequency should be discussed at the introductory level.

Keywords: wave property, frequency, phase velocity

1. Introduction

The time-evolution of electrons and other 'sub-atomic' particles is described by the laws of quantum mechanics rather than classical (Newtonian) mechanics. The history of quantum mechanics is fascinating, and we refer the reader to Pais' authoritative account [1]. An electron can interfere with itself and exhibit wave-like properties.

Wave-like properties of electrons are emphasized in texts for the introductory sequence [2] and the sophomore-level modern physics course [3–5]. They are based on the particle–wave dualism, which has its origin in Einstein's explanation of the photoeffect. Light (electromagnetic waves) with frequency ν and wavelength λ (such that $c = \lambda\nu$ is the speed of light) has particle-like properties, i.e., monochromatic light can be described by a stream of photons with energy $E = h\nu$ and momentum $p = h/\lambda$.

The momentum–wavelength relationship is used to describe *standing* electron waves from which the energy of an electron in a 'box' or along a (circular) orbit is derived. In contrast, the energy–frequency relationship $\nu = E/h$ for the electron is mostly ignored in introductory discussions. This curious 'asymmetry' between the wavelength and frequency of an electron wave was noted by an inquisitive student in my (algebra-based) introductory physics course: she insisted that any wave-like behavior implies periodic behavior in both space *and* time. For a classical wave, the product of frequency and wavelength defines the (phase) speed, which suggests that the frequency of an electron wave captures some aspects of quantum dynamics.

I had the good fortune to first consult the text by Eisberg and Resnick [3] and the Feynman lectures [4] before I answered the student. To my surprise, these two standard texts give two different answers for the electron frequency based on the non-relativistic and relativistic expression for the electron energy, respectively. A subsequent search of standard texts showed that the authors choose either the relativistic or non-relativistic expression for the frequency of the electron wave *without* mentioning that the other expression is given in the literature. I found this to be quite startling since physical quantities are supposed to have well-defined values.

While the wave function is not an *observable* in quantum mechanics, it is the solution of a partial differential equation (subject to suitable boundary conditions). Thus, $\psi(x, t)$ is well-defined for an initial condition, $\psi(x, 0)$. In this paper, we use the terms 'wavelength' and 'frequency' to mean quantities characterizing the spatial and temporal periodicity of an electron wave associated with a unique solution of the Schrödinger equation. That is, the time evolution of the wave function defines a unique frequency of an electron wave.

The relativistic and non-relativistic expression for the energy only differ by a constant term, namely the electron rest energy. Absolute energy values are irrelevant in classical mechanics since only energy differences are useful, and the energy E of a system can be changed $E \to E + E_0$ for an arbitrary value of E_0. In contrast, the frequency of a wave is determined by the frequency of the periodic motion of its source; thus the frequency of classical wave from a single source has a unique value that does not change even if the phase speed changes. These two properties of classical systems are incompatible with the frequency–energy relationship $\nu = E/h$: (1) an electron wave with a unique frequency implies a well-defined value for the classical electron energy, (2) if the classical electron energy is determined up to a constant only, the frequency of the electron wave does not have a unique value.

I found the choices (1) and (2) to be rather 'unappealing', since they violate basic properties of a classical particle or a classical wave. This shows that the application of the energy–frequency correspondence to particle waves may not be as straightforward as it may seem. The correspondence principle states that there is, of course, a quantum mechanical system that reduces to a classical electron traveling at a constant velocity v. We find that the difficulty determining the frequency of the electron wave reflects the 'strangeness' of the quantum-mechanical 'world' of particles [6]. While $\psi(x, t) = \exp[\mathrm{i}\hbar^{-1}(px - Et)]$ with $E = p^2/2m$ is indeed the solution of the Schrödinger equation for a free particle in one dimension, we show that the solution cannot be interpreted as a propagating wave in the classical sense.

For classical systems, the disturbance of a wave is caused by a source, which is a harmonic oscillator in the case of a plane wave. If we hypothesize that a particle wave also has a source, the title of the paper raises the question what physical system we should identify as the source of an electron wave.

The outline of the paper is as follows. In section 2, we sketch a derivation of the relativistic and non-relativistic expressions for the frequency and the phase velocity of the electron wave; the expressions are familiar and more detailed derivations can easily be found. We review the literature in section 3. We discuss the electron wave in one spatial dimension in section 4 since no additional insight is obtained by considering higher dimensions. We show in particular that an energy shift would correspond to an additional term in the

Schrödinger equation. In section 5 we review de Broglie's thesis and explain that the frequency must be based on the relativistic expression for the energy. We also discuss how I answered the student. We conclude with a summary in section 6.

2. Phase and group velocity

A monochromatic classical wave $u(x, t)$ is a disturbance that is periodic in space with wavelength λ, $u(x + \lambda, t) = u(x, t)$ and periodic in time with period T, $u(x, t + T) = u(x, t)$. The wave number is defined in terms of the wavelength, $k = 2\pi/\lambda$, and the (angular) frequency is defined in terms of the period, $\omega = 2\pi/T$. A propagating wave depends on the position and time through the combination $x \pm v_p t$ or $z = kx \pm \omega t$ so that $u(x, t) = u(kx \pm \omega t)$, where the sign '+' ('−') describes a wave traveling in the negative (positive) x direction.

The velocity of a crest, or trough, is defined by the condition $\mathrm{d}z = k\mathrm{d}x \pm \omega \mathrm{d}t = 0$, and one finds the phase velocity $v_p = \omega/k$. The center of a localized disturbance travels at the group velocity [7], $v_g = \mathrm{d}\omega/\mathrm{d}k$. The wave is dispersive when the phase velocity depends on the wave number, $v_p = v_p(k)$. We write $\omega = v_p k$ so that $v_g = v_p + k\mathrm{d}v_p/\mathrm{d}k$. If the wave is dispersion-free, $\mathrm{d}v_p/\mathrm{d}k = 0$, and the phase and group velocities are identical $v_p = v_g$.

For an electron wave, we start from the expressions between wave- and particle-like properties, $\omega = E/\hbar$ and $k = p/\hbar$. The phase velocity of an electron wave then follows as the ratio of electron energy and momentum: $v_p = E/p$, and the group velocity is the derivative of the energy with respect to the momentum, $v_g = \mathrm{d}E/\mathrm{d}p$. We find very different results whether the non-relativistic or relativistic expression for the electron energy is used.

We start from the non-relativistic expression for the kinetic energy $E_{nr} = p^2/2m$ so that the electron frequency follows

$$\omega_{nr} = \frac{p^2}{2\,m\hbar} \quad \text{(non-relativistic)} \tag{1}$$

and the phase velocity is given by $v_p = (p^2/2m)/p = p/2m$, or

$$v_{p,nr} = \frac{v}{2} \quad \text{(non-relativistic)} \tag{2}$$

and the group velocity $v_g = 2p/2m = v$. For a relativistic particle, the relationship between energy and momentum is given by $E_r^2 - (pc)^2 = m^2c^4$. At low speeds $v < c$, the electron energy is dominated by its rest energy $E_r \simeq E_0 = mc^2$ so that the frequency follows

$$\omega_r \simeq \omega_0 = \frac{mc^2}{\hbar} \quad \text{(relativistic).} \tag{3}$$

We find the phase velocity $v_p = mc^2/p = c^2/(p/m)$ so that

$$v_{p,r} \simeq \frac{c^2}{v} \quad \text{(relativistic).} \tag{4}$$

We have $2E\mathrm{d}E - 2pc^2\mathrm{d}p = 0$ so that $v_g = (2pc^2)/2E = v$. We note that the group velocity is equal to the velocity of an electron for both the non-relativistic and relativistic expressions. We conclude that the correspondence principle is no help in determining whether we should choose the relativistic or the non-relativistic expression for the frequency of the electron wave.

Since there is no ambiguity regarding the wavelength of the electron, two different values for the frequency will lead to two different values of the phase velocity. In the non-relativistic case, the phase speed is half the velocity of the particle. In the relativistic case, the phase velocity is greater than the speed of light. Superluminal phase velocities are also known in classical electrodynamics [8]; it is not a violation of fundamental laws of physics since signals are not propagated by the phase velocity, but rather by the group velocity that is always less than the speed of light.

As an instructive numerical example, we calculate the frequency of an electron that is accelerated through a 1 V electrostatic potential. The final kinetic energy is $E = p^2/2m = 1.60 \times 10^{-19}$ J so that the speed follows $v = \sqrt{2E/m} \simeq 5.92 \times 10^5$ m s^{-1}, and the ratio of the electron speed to the speed of light follows $v/c \simeq 2.0 \times 10^{-3}$. The frequency in the non-relativistic limit follows

$$\omega_{\rm nr} = \frac{1.60 \times 10^{-19}\ \text{J}}{1.05 \times 10^{-34}\ \text{J s}} = 1.52 \times 10^{15}\ \text{s}^{-1}. \tag{5}$$

The rest energy of the electron is $mc^2 = 511$ keV $= 8.20 \times 10^{-14}$ J. The frequency for the relativistic version follows, $\omega_{\rm r} \simeq \omega_0$ with

$$\omega_0 = \frac{8.20 \times 10^{-14}\ \text{J}}{1.05 \times 10^{-34}\ \text{J s}} = 7.81 \times 10^{20}\ \text{s}^{-1}. \tag{6}$$

We thus see that the non-relativistic and relativistic expressions yield quite different numerical values for the frequency of the electron wave. This is expected. The ratio of the frequencies follows $\omega_0/\omega_{\rm nr} \simeq 5 \times 10^5$. However, the fact that the difference is so large is not necessarily expected.

3. Review of literature

In this section we make an attempt to include standard texts as well as more obscure ones, although we do not claim that our literature search is exhaustive. We first review texts that start from the relativistic expression of the energy and then discuss texts that instead start from the non-relativistic expression.

Born writes [9] 'the phase velocity is given by [$v_{\rm p} = c^2/v$] and is therefore greater than the velocity of light c, if the particle's velocity is less than $v < c$. The phases of the matter wave are therefore propagated with a velocity exceeding that of light'. Pauli also stresses the relativistic dynamics [10, 11]: 'the idea of de Broglie was that [...] should also be valid for a material particle in which case [...] $(\omega/c)^2 = \vec{k}^2 + m^2c^2/h^2$. [...] From [...] we also obtain $\omega = E/\hbar = mc^2/\hbar + \hbar k^2/2m + ... = mc^2/\hbar + \omega'$. [...] From now on we shall always calculate with the primed quantities which we have introduced here; however, for the sake of simplicity, the primes will be left off. The quantities ω and ω' only differ by a constant; however, this is not an essential difference since only frequency differences are ever of importance in wave mechanics'.

Messiah wrote the leading French textbook on quantum mechanics [12], which presumably reflects de Broglie's thinking. He uses the relativistic expressions for the frequency and phase velocity. The fact that the latter is greater than the speed of light $v_{\rm p} > c$ is noted, but is not discussed in any detail. The relativistic expression of the energy is used in both the Feynman lectures [4] and the Berkeley lectures [13]. The relativistic four-vector particle–wave correspondence $(\vec{k}, \omega) = \hbar^{-1}(\vec{p}, E)$ is emphasized in the recent text by Weinberg [14].

The non-relativistic expression of the energy is used in introductory texts [2, 3, 5], as well as the textbooks by Griffiths [15], Sakurai [16], Cohen-Tannoudji *et al* [17], and Liboff [18]. The reasoning is based on the solution of the Schrödinger equation in one dimension.

Gasiorowicz [19] starts from the time evolution of wave packets, and then discusses the time-dependence of the wave function: 'The solution of [the one-dimensional Schrödinger equation] is determined by $\psi(x, 0)$. This is in contrast to the familiar wave equation $\partial^2 f/\partial t^2 = v_p^2 \partial^2 f/\partial x^2$ in which both $f(x, 0)$ and $\partial f(x, t)/\partial t|_{t=0}$ have to be specified[1]. The difference is a consequence of the fact that the Schrödinger equation is first order in [time] t. We shall see that this is closely related to the probability interpretation of $\psi(x, t)$'. The discussion in Merzbacher [21] mirrors the treatment by Pauli. He starts from the equation for the group velocity and finds that the electron frequency is only determined up to a constant. Merzbacher then explains that the unknown value of the frequency of a non-relativistic electron reflects the fact that absolute values of energy are meaningless in non-relativistic mechanics. Essentially the same reasoning is found in the graduate-level text by Böhm [22]. Baym starts from the propagation of a wave packet and arrives at the non-relativistic expression for the phase velocity [23].

To summarize, older texts tend to give the relativistic expressions for the frequency and phase speed of electrons, whereas more recent texts tend to favor the corresponding non-relativistic expressions. Older texts acknowledge that the relativistic phase speed is greater than the speed of light; they attribute this unphysical speed as an indicator that the wave function is not an observable in quantum mechanics.

The somewhat obscure text by Kramers [24] is an exception insofar that it discusses both the relativistic and non-relativistic expression for the electron frequency. He notes that interference experiments cannot confirm the correctness of either relation since only interference in space, but not in time, can be measured. Kramers studies the equation of continuity for the probability current $\vec{j}$: $\oint \vec{j} \cdot \mathrm{d}\vec{A} = -(\mathrm{d}/\mathrm{d}t) \int |\psi|^2 \mathrm{d}V$, and derives $j_x = (2\pi \mathrm{i})^{-1} [\mathrm{d}\omega/\mathrm{d}(k^2)] \cdot [\psi^*(\mathrm{d}\psi/\mathrm{d}x) - \psi(\mathrm{d}\psi^*/\mathrm{d}x)]$. He examines both the relativistic and non-relativistic dispersion laws: he finds that only the non-relativistic version obeys the conservation law for particle number, whereas the relativistic dispersion law, 'which we should really have used as the basis of our theory would have led to great difficulties'. On the other hand, Kramers argues that matter waves should be invariant under a Lorentz transformation. He writes that 'this difficulty is real. It is the first indication in our approach to the theory that the foundations which we have been using are insufficient to extend the theory so as to incorporate the requirements of the theory of relativity. These difficulties are surmounted to a large extent in Dirac's theory of the spinning electron'.

4. Time-dependent Schrödinger equation

The view of an electron as a propagating wave emerges from the solution of the one-dimensional Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}. \tag{7}$$

The solution is well-known

$$\psi(x, t) = \mathrm{e}^{\mathrm{i}\hbar^{-1}(px - E_{\mathrm{nr}} t)} \tag{8}$$

with the non-relativistic expression for the electron energy, $E_{\mathrm{nr}} = p^2/2m$. While this solution has the mathematical form of a propagating plane wave, its interpretation in classical and quantum physics is different. We recall that the exponential function has a real and imaginary

[1] We assume that the wave is dispersion-free; see [20].

part, $\exp(\mathrm{i}z) = \cos(z) + \mathrm{i}\sin(z)$. In classical physics, the real and imaginary parts are two linearly independent solutions in the form of propagating waves $u_1(x, t) = \cos(kx - \omega t)$ and $u_2(x, t) = \sin(kx - \omega t)$. This is not the case for the solution of the Schrödinger equation. We use column vector notation for complex numbers

$$\psi = \begin{pmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{pmatrix} = \begin{pmatrix} \cos(\hbar^{-1}[px - E_{\mathrm{nr}}t]) \\ \sin(\hbar^{-1}[px - E_{\mathrm{nr}}t]) \end{pmatrix} \tag{9}$$

and write the Schrödinger equation separately for the real and imaginary parts: $-\hbar\partial\psi_2\partial t = -(\hbar^2/2m)\partial^2\psi_1/\partial x^2$ and $\hbar\partial\psi_1/\partial t = -(\hbar^2/2m)\partial^2\psi_2/\partial x^2$, or as coupled equations

$$\hbar\frac{\partial}{\partial t}\begin{pmatrix} -\psi_2(x, t) \\ \psi_1(x, t) \end{pmatrix} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\begin{pmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{pmatrix}. \tag{10}$$

That is, the real and imaginary parts of $\exp[\mathrm{i}\hbar^{-1}(px - E_{\mathrm{nr}}t)]$ are *not* two independent solutions of the Schrödinger equation. In particular, the ratio E_{nr}/p must not be interpreted as a phase velocity in the classical sense. Indeed, the usual meaning of the term phase velocity means that any functional form $u(x - v_{\mathrm{p}}t)$ is a solution of a dispersion-free wave equation. A propagating electron wave does *not* have this property since the Schrödinger equation is not second order in both space and time, and thus the only functional form for $\psi(x, t)$ for a free particle is the complex exponential.

The absolute value of energy has no meaning in classical, non-relativistic physics. So the change $E_{\mathrm{nr}} \to E_{\mathrm{nr}} + E_0$ with E_0 does not change the particle dynamics. For an electron wave, a change in the energy corresponds to a phase change $\psi(x, t) \to \psi'(x, t) = \mathrm{e}^{-\mathrm{i}\hbar^{-1}E_0 t}\psi(x, t)$ that does not change expectation values and probabilities (that is, observables in quantum mechanics). However, the wave functions $\psi(x, t)$ and $\psi'(x, t)$ are *not* two solutions of the same Schrödinger equation; rather the equation for $\psi'(x, t)$ has an extra term

$$\mathrm{i}\hbar\frac{\partial\psi'}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi'}{\partial x^2} + E_0\psi'. \tag{11}$$

Thus, an energy shift requires a potential energy-like term $E_0\psi(x, t)$ in the Schrödinger equation. For a non-relativistic 1 eV electron, $E_{\mathrm{r}} \simeq E_0 + E_{\mathrm{nr}}$, this shift is equal to the electron rest energy $E_0 = mc^2$. The potential energy term in equation (11) reflects a property of the electron itself rather than external environment the electron finds itself in.

5. de Broglie thesis

We only sketch the salient points in de Broglie's thesis [25]. His starting point is the energy–frequency relationship: energy $\simeq \hbar \times$ (angular) frequency, where the energy is its value in the rest-frame of the electron. That is, de Broglie assumes the relativistic expression for the frequency of the electron wave, $\omega_0 = mc^2/\hbar$. It should be noted that de Broglie's theory is consistent with the view that the electron acts as its own source.

If the electron moves with a speed v relative to a laboratory frame (x, t), the Lorentz transformation yields the expression for the proper time of the electron

$$t_0 = \frac{1}{\sqrt{1 - (v/c)^2}}\left(t - \frac{vx}{c^2}\right). \tag{12}$$

de Broglie assumes that the electron wave function oscillates in the electron rest frame

$$\psi(t_0) = \sin(\omega_0[t_0 - \tau_0]), \tag{13}$$

where τ_0 is a constant. It follows for an observer in the laboratory frame (x, t), $\psi(x, t) = \sin\{\omega_0/\sqrt{1-(v/c)^2} \cdot [t - (vx/c^2) - \tau_0]\}$, or

$$\psi(x, t) = \sin\left\{\frac{mv}{\sqrt{1-(v/c)^2}\,\hbar}\left[x - \frac{c^2}{v}(t - \tau_0)\right]\right\}. \tag{14}$$

While this expression 'looks like' the equation for a traveling wave with phase speed $v_{\mathrm{p,r}} = c^2/v$, it is based entirely on the appropriate Lorentz transformation and has also been explained within the context of clock desynchronization [26].

de Broglie discusses that the wave speed c^2/v implied by equation (9) is greater than the speed of light, and thus cannot represent the transport of energy. He suggests that 'this wave represents a spatial distribution of *phase*, that is to say, it is a "phase wave"'. de Broglie does not mention the momentum–wavelength relation in his thesis at all, but discusses it in his 1929 Nobel lecture [27]. He writes $p = mv/\sqrt{1-(v/c)^2}$ for the relativistic momentum of the electron, and adds that 'this is a fundamental relation of the theory'.

It is left to historians of science to try to trace de Broglie's physical reasoning towards the development of his theory. It would be particularly interesting to know whether he viewed the energy–frequency and momentum–wavelength relationships on equal footing, or whether he viewed one relationship as more fundamental than the other. Unfortunately, Pais is silent about this point [1]. We find it quite striking, though, that the starting point of de Broglie's thesis is the energy–frequency relation, whereas introductory texts seem to suggest that the momentum–wavelength relation is the fundamental expression for the wave nature of electrons.

We now return to the question posed in the title of the paper. The fundamental and correct theory is relativistic expression equation (3) $\omega_{\mathrm{r}} \simeq \omega_0 = mc^2/\hbar$ and the apparent ambiguity of the frequency only arises if one considers the non-relativistic approximation of the relativistic Klein–Gordon equation for an electron without spin[2].

However, this is not how I can answer my student. A teacher must find a way to explain concepts at a level appropriate for that student, which can be particularly challenging at the introductory level. I emphasize that the notion of an electron *wave*, as opposed to the electron wave function, should only be considered as a classical analog to 'true' quantum behavior. I chose to follow de Broglie's original idea and explained that the dominant portion of the electron frequency is determined by the rest energy $E_0 = mc^2$ and that the (non-relativistic) kinetic energy $E_{\mathrm{nr}} = p^2/2m$ induces a frequency modulation

$$\omega = \omega_0 + \omega_{\mathrm{nr}}. \tag{15}$$

I use the analogy of frequency modulation of radio waves to answer my student. In equation (15) ω_0 is the analog to the frequency of the radio station, e.g., 90.3 MHz for the local public radio station in Cleveland, OH, and ω_{nr} is the analog of the frequency of the music played on the radio (which is then transformed into sound sound waves by loudspeakers). We use the frequency for the musical tone A and find the ratio $90.3\ \mathrm{MHz}/440\ \mathrm{Hz} \simeq 2 \times 10^5$, which accidentally turns to be comparable to the ratio of electron frequencies, $\omega_0/\omega_{\mathrm{nr}}$.

[2] A brief discussion can be found in [23, chapter 22].

6. Summary

Introductory texts discuss the momentum–wavelength duality of particles and waves but are (mostly) silent about the energy–frequency relationship. My student was correct to point that this explanation is unsatisfactory since (propagating) waves are periodic both in space and time, and a pane wave must have a source that oscillates with a well-defined frequency.

In de Broglie's theory, the electron in its rest frame is the oscillating system and in this sense acts as its own source. The electron frequency is determined by the relativistic expression of the energy $E_{\mathrm{r}} \simeq mc^2$ together with the energy–frequency relation $\omega_{\mathrm{r}} = E_{\mathrm{r}}/\hbar$. Several current texts fail to make the connection to the underlying relativistic theory and exclusively base their discussion of the electron frequency on the non-relativistic Schrödinger equation. Our discussion shows that the notion of 'electron wave' is flawed when it is used in a literal description within a classical context.

Textbooks should emphasize more that the electron cannot be described entirely in terms of classical concepts and that, in particular, the model of a propagating phase should only be considered as a classical analog. With this caveat in mind, our discussion suggests that the best answer regarding the frequency of an electron wave is the relativistic expression $\omega_0 = mc^2/\hbar$. This choice was also favored by the founders of quantum mechanics (de Broglie, Born, etc) but somehow got 'lost' along the way.

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