TFY4345 Classical Mechanics Exam November 27, 2023

Part I (2.5 points for each correct answer. Answer Part I in the table in Inspera.)

1.1 The constraint $r = \sqrt{x^2 + y^2}$ for a particle sliding on a ring of radius r is called

A) conservative B) holonomic C) canonical D) invariant E) cyclic F) virtual

1.2 What is the value of the element ε_{321} of the Levi-Civita tensor?

A) 1 B) -1 C) i D) $-i$ E) 0 F) π

1.3 What is the SI unit of the conjugate momentum to the polar angle θ?

A) The same unit as for angular momentum
B) The same unit as for power
C) The same unit as for acceleration
D) Nm
E) m/s^2
F) kg m/s

1.4 If the system Lagrangian is independent of a coordinate q, this coordinate is called

A) conservative B) holonomic C) canonical D) invariant E) cyclic F) virtual

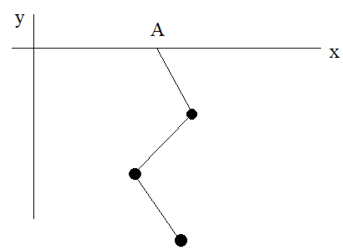
1.5 A particle with mass m moves in the xy plane in a potential $V(r) = kr^2/2$. Here, k is a positive constant, $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\sin\theta$. Which quantity (in addition to the total energy) is conserved for this particle?

A) p_x B) p_y C) v_x D) v_y E) p_θ F) None

1.6 Three point particles moving in three dimensional space are subject to three independent holonomic constraints. How many independent coordinates q_j are needed to describe this system?

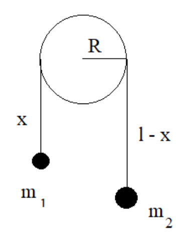
A) 3 B) 4 C) 5 D) 6 E) 7 F) 8

1.7 A triple planar pendulum consists of three balls (point masses) connected by two massless rods of fixed length, and by a third massless rod to the support at A, which may slide without friction in the x direction. The balls are allowed to move in the xy plane. How many independent coordinates q_j are needed to describe this system?



A) 3 B) 4 C) 5 D) 6 E) 7 F) 8

1.8 Two balls (point masses m_1 and m_2) are connected with a massless rope of length $\ell + \pi R$ that can slide without friction over a cylinder with radius R. Zero potential is chosen at the centre of the cylinder, a distance x above m_1. What is the Lagrangian $L = T - V$ for this system?



A) $L = \frac{1}{2}(m_1 - m_2)\dot{x}^2 + (m_1 - m_2)gx + m_2 g\ell$

B) $L = \frac{1}{2}(m_1 - m_2)\dot{x}^2 + (m_1 + m_2)gx + m_2 g\ell$

C) $L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx + m_2 g\ell$

D) $L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 + m_2)gx + m_2 g\ell$

E) $L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx + (m_1 + m_2)g\ell$

F) $L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx + (m_1 - m_2)g\ell$

1.9 A particle with mass m and charge q moving in an electromagnetic field has the Lagrangian

$$L = \frac{1}{2} m \dot{x}_j \dot{x}_j + q A_j \dot{x}_j - q\phi.$$

What is the canonical momentum p_2?

A) $p_2 = q\dot{x}_2 + mA_2$
B) $p_2 = m\dot{x}_1 - qA_3$
C) $p_2 = \dot{x}_2 - A_2$
D) $p_2 = 2m\dot{x}_2 + qA_2$
E) $p_2 = 2m\dot{x}_2 + qA_2/2$
F) $p_2 = m\dot{x}_2 + qA_2$

1.10 If the vector potential is $\boldsymbol{A} = B_0(x\hat{y} - y\hat{x})$, what is the magnetic field $\boldsymbol{B}$?

A) $\boldsymbol{B} = B_0\hat{z}$ B) $\boldsymbol{B} = 2B_0\hat{z}$ C) $\boldsymbol{B} = (B_0/2)\hat{z}$
D) $\boldsymbol{B} = 4B_0\hat{z}$ E) $\boldsymbol{B} = (B_0/4)\hat{z}$ F) $\boldsymbol{B} = 0$

1.11 If the vector potential is $\boldsymbol{A} = E_0 t\hat{x}$ and the scalar potential is $\phi = E_0(y+z)$, what is the electric field $\boldsymbol{E}$?

A) $\boldsymbol{E} = -E_0\hat{x}$ B) $\boldsymbol{E} = -E_0(\hat{y}+\hat{z})$ C) $\boldsymbol{E} = -E_0(\hat{x}+\hat{z})$
D) $\boldsymbol{E} = -E_0(\hat{x}+\hat{y})$ E) $\boldsymbol{E} = -E_0(\hat{x}+\hat{y}+\hat{z})$ F) $\boldsymbol{E} = 0$

1.12 If the Lagrangian for a system with two independent coordinates q_1 and q_2 is

$$L = c_1 q_1^2 + c_2 q_2^2 + c_3 q_1 \dot{q}_1 + c_4 \dot{q}_2^2,$$

what is the canonical momentum p_1?

A) $p_1 = c_1 q_1$ B) $p_1 = c_2 q_1$ C) $p_1 = c_3 q_1$
D) $p_1 = c_1 q_2$ E) $p_1 = c_2 q_2$ F) $p_1 = c_3 q_2$

1.13 A planet moves in an elliptical orbit with eccentricity 0.21 around a much heavier star located at the origin. What is the ratio $r_{\rm min}/r_{\rm max}$ between the shortest and longest distance from the planet to the star?

A) 0.45 B) 0.55 C) 0.65 D) 0.75 E) 0.85 F) 0.95

1.14 Which transformation matrix describes rotation an angle ϕ counterclockwise around the x_2 axis?

$$\text{A)} \quad \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$\text{B)} \quad \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & -\cos\phi \end{pmatrix}$$

$$\text{C)} \quad \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$\text{D)} \quad \begin{pmatrix} \sin\phi & 0 & -\cos\phi \\ 0 & 1 & 0 \\ \cos\phi & 0 & \sin\phi \end{pmatrix}$$

$$\text{E)} \quad \begin{pmatrix} -\sin\phi & 0 & -\cos\phi \\ 0 & 1 & 0 \\ \cos\phi & 0 & \sin\phi \end{pmatrix}$$

$$\text{F)} \quad \begin{pmatrix} -\cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & -\cos\phi \end{pmatrix}$$

1.15 A spaceship moves with speed $7c/8$ relative to Spaceman Spiff, who is at rest on Anhooie-4. The alien Hideous Blob is shot out of the spaceship, in the forward direction, with speed $5c/6$ relative to the spaceship. What is the speed of Hideous Blob as observed by Spaceman Spiff?

A) $82c/83$ B) $6c/7$ C) $41c/24$ D) $7c/8$ E) $35c/48$ F) $c/3$

1.16 The acceleration $\boldsymbol{a}_b$ of an object measured by a stationary observer at the Equator on the surface of the Earth is

$$\boldsymbol{a}_b = \boldsymbol{a}_s + 2\boldsymbol{v}_b \times \boldsymbol{\omega} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}).$$

Here, $\boldsymbol{v}_b$ and $\boldsymbol{r}$ are the velocity and the position of the object, respectively, both measured by this observer, and $\boldsymbol{a}_s$ is the acceleration measured in an inertial system. Assume a spherical Earth with radius $r = 6378$ km rotating around an axis pointing straight north, with a period $T = 2\pi/\omega = 24$ hours. The observer throws a ball upwards with an initial speed 15 m/s. At this instant, what is the Coriolis acceleration?

A) 5.5 mm/s^2 B) 4.4 mm/s^2 C) 3.3 mm/s^2 D) 2.2 mm/s^2 E) 1.1 mm/s^2 F) 0.55 mm/s^2

1.17 At the instant described in the previous question, what is the centrifugal acceleration?

A) 64 mm/s^2 B) 54 mm/s^2 C) 44 mm/s^2 D) 34 mm/s^2 E) 24 mm/s^2 F) 14 mm/s^2

1.18 What is the mass of a (free) particle with energy 500 MeV and momentum 400 MeV/c?

A) 50 MeV/c^2 B) 100 MeV/c^2 C) 150 MeV/c^2
D) 200 MeV/c^2 E) 250 MeV/c^2 F) 300 MeV/c^2

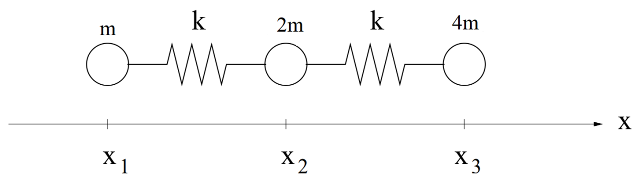
1.19 In a canonical transformation of type 1, from "old" coordinates (q, p) to "new" coordinates (Q, P), the generating function is $F = F_1(q, Q) = qQ - q^2 - Q^2$. What is $P(q, p)$?

A) $P = q - p$ B) $P = 4q + 3p$ C) $P = q + p$
D) $P = 2q - 2p$ E) $P = 4q - 3p$ F) $P = 3q + 2p$

1.20 What is most likely a key ingredient in Unni's Cinnamon Cake?

A) Anise
B) Basil
C) Cinnamon
D) Dill
E) Estragon
F) Fennel

Part II (Weights are given for each of the 8 partial questions)



2 (20%) Three balls with masses m, $2m$ and $4m$ ($m = 50$ g) are connected by identical and ideal springs with spring constant $k = 150$ N/m, as shown in the figure above. The balls can move along the x axis only, and we consider small oscillations around their equilibrium positions x_{01}, x_{02} and x_{03}.

a) (7%) With the balls' deviations from equilibrium, $\eta_i = x_i - x_{0i}$, as coordinates, the potential V and kinetic energy T are both quadratic forms,

$$V = \frac{1}{2} V_{ij} \eta_i \eta_j \quad \text{and} \quad T = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j ,$$

respectively. Determine the 3×3 symmetric matrix $\boldsymbol{V}$ and diagonal matrix $\boldsymbol{T}$, with matrix elements V_{ij} and T_{ij}.

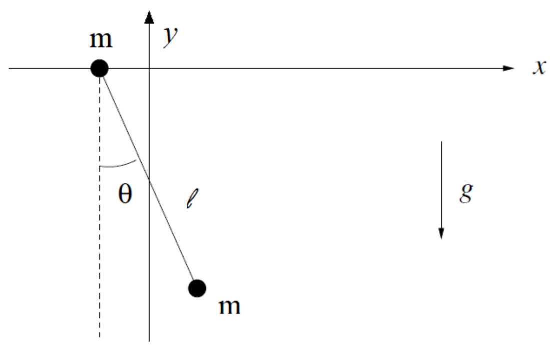
b) (7%) Solve the secular equation

$$\left| \boldsymbol{V} - \omega^2 \boldsymbol{T} \right| = 0$$

(i.e., zero determinant) and determine the two nonzero eigenfrequencies $f_j = \omega_j / 2\pi$ ($j = 1, 2$) of this system. (Determine both numerical values and units.)
Hint: You will end up with a 3rd order equation for ω^2, where one root is $\omega^2 = 0$. You may find it convenient to extract a factor k^3 from the determinant and introduce the dimensionless variable $\alpha = m\omega^2/k$.

c) (6%) Determine the amplitudes (i.e., relative amplitudes, including sign) of the three balls in the normal mode with the *smallest* eigenfrequency.

3 (25%) A rod with negligible mass and length ℓ has equal masses m at its two ends. (See figure below.) One mass can slide without friction along a horizontal constraint on the x axis. The other mass is restricted to move in the xy plane. We consider a situation where the center of mass is all the time located on the y axis, i.e., in $x = 0$. Zero potential energy is chosen in vertical position $y = 0$.



a) (7%) Show that the Lagrangian of the system is

$$L(\theta, \dot{\theta}) = T - V = \frac{m\ell^2\dot{\theta}^2}{4}(1 + \sin^2\theta) + mg\ell\cos\theta.$$

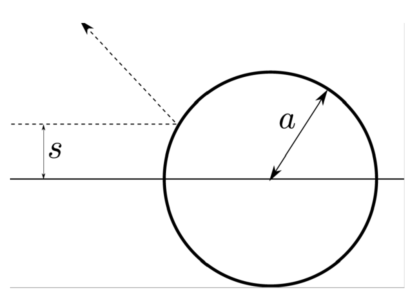
Hint: $a\sin^2 x + b\cos^2 x = b + (a - b)\sin^2 x$.

b) (7%) Find the equation of motion (i.e., the Lagrange equation).

c) (5%) If the oscillation amplitude is small, the system is a harmonic oscillator. Show this by including only linear terms in the equation of motion, and determine the oscillation frequency ω.

d) (6%) Find the Hamiltonian $H = p_\theta\dot{\theta} - L$ expressed as a function of the canonical variables θ and p_θ.

4 (5%) A point particle moving in two dimensions collides with a hard disk with radius a. If the impact parameter is $s = a/\sqrt{2}$, what is the scattering angle θ?
The general situation is illustrated in the figure below. The scattering angle θ is defined as the angle between the point particle's incoming and outgoing direction.



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FORMULAS FOR EXAM 27.11.2023

The meaning of the symbols is assumed to be known.

$$L = T - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

$$H = \sum_i p_i \dot{q}_i - L$$

$$p = \frac{\partial L}{\partial \dot{q}}$$

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}$$

$$r = \frac{p}{1 + \varepsilon \cos\theta}$$

$$\boldsymbol{A}_{\text{rot}} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$x' = x \ , \ y' = y \ , \ z' = \gamma(z - vt) \ , \ t' = \gamma(t - vz/c^2) \ , \ \gamma = (1 - v^2/c^2)^{-1/2}$$

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix} \quad , \quad \beta = v/c$$

$$v_{31} = \frac{v_{32} + v_{21}}{1 + v_{32}v_{21}/c^2}$$

$$E^2 = (pc)^2 + (mc^2)^2$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad , \quad \boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$F = F_1(q, Q) \ , \ p = \frac{\partial F_1}{\partial q} \ , \ P = -\frac{\partial F_1}{\partial Q} \ , \ K = H$$

$$[u, v] = \frac{\partial u}{\partial q}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial q}$$