NORGES TEKNISK-
NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR ENERGI- OG PROSESSTEKNIKK

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EXAM TEP4145 KLASSISK MEKANIKK
Thursday 16. August 2007 kl. 1200 - 1600
English version

Remedies: C

- K. Rottmann: Mathematical formulae
- Approved calculator, with empty memory, according to list worked out by NTNU. (HP30S or similar.)

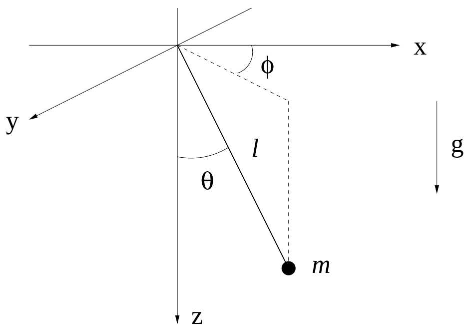
Page 2 - 6: The questions
Page 7: Some formulas

The exam consists of 5 questions. The weight for each question is given. Do questions 1, 2 and 3, and either question 4 (physics) or 5 (cybernetics).
Note that Einstein's summation convention is used unless otherwise stated, i.e., summation over repeated indices.

The grades will be ready in August.

QUESTION 1 [Counts 30%]

A spherical pendulum consists of a mass m attached to the end of a massless rod of length l. The other end of the rod is attached at the origin. In other words, the mass may move on a spherical shell of radius l, hence the name. The system is in the gravitational field.



a) Show that the Lagrangian of the system is

$$L = \frac{ml^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + mgl\cos\theta$$

when we choose $V = 0$ in $z = 0$, and use generalized coordinates θ and ϕ as shown in the figure.

b) Explain how you immediately can say that the canonical momentum p_ϕ is a constant of the motion. Show that p_ϕ is identical to the z-component of the angular momentum, i.e., $(\boldsymbol{r} \times m\boldsymbol{v})_z$.

c) Introduce the constant

$$\gamma = \dot{\phi}\sin^2\theta = \frac{p_\phi}{ml^2}$$

and show that Lagrange's equation for the coordinate θ becomes

$$\ddot{\theta} + \frac{g}{l}\sin\theta - \gamma^2\frac{\cos\theta}{\sin^3\theta} = 0$$

d) One possibility is that the mass moves in a horizontal circle, at constant $\theta = \theta_0$. Show that the period (time of revolution) then becomes

$$T_\phi = 2\pi\sqrt{\frac{l\cos\theta_0}{g}}$$

e) Another possibility is that the mass "oscillates" around θ_0, i.e., $\theta(t) = \theta_0 + \alpha(t)$. Assume small oscillations, i.e. $|\alpha| \ll \theta_0$ (and hence $|\alpha| \ll 1$), and show that the motion of the pendulum is described by the equation

$$\ddot{\alpha} + \alpha\,\frac{g}{l}\cos\theta_0\left(4 + \tan^2\theta_0\right) = 0$$

f) What is the oscillation period T_α? Calculate the ratio between the time of revolution and the oscillation period, i.e., T_ϕ/T_α, and sketch this ratio as a function of the angle θ_0.

QUESTION 2 [Counts 30%]

A particle of mass m moves in a spherically symmetric potential $V(r)$. It can be shown that the motion is in a plane perpendicular to the direction of the angular momentum $\boldsymbol{L}$, which is a constant of the motion, with $|\boldsymbol{L}| = |\boldsymbol{r} \times \boldsymbol{p}| = l$.

a) Determine the Lagrangian $L = T - V$ of the particle, using the polar coordinates r and θ as generalized coordinates. (Don't confuse the Lagrangian L with the angular momentum $\boldsymbol{L}$.)

b) Show that Lagrange's equation for θ becomes

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0.$$

Show that this expresses conservation of the angular momentum.

c) Show that Lagrange's equation for r becomes

$$m\ddot{r} - mr\dot{\theta}^2 = f(r),$$

where $f(r) = -\partial V/\partial r$ is the force acting on the particle.

d) Use the equations and the information given above to derive the following differential equation for $u = 1/r$:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2}\, f(1/u)$$

e) If the particle moves in a $1/r$-potential, the path will be described by the conic section

$$r(\theta) = \frac{\alpha}{1 + \varepsilon\cos\theta},$$

where α and ε are constants. Use the differential equation of the orbit in *d*) to verify this. (Assume $V(\infty) = 0$.)

f) Next, assume that the particle follows the spiral orbit

$$r(\theta) = r_0 e^{\theta},$$

where r_0 is a constant. Find the potential $V(r)$ in which this particle moves.
(Also here, assume $V(\infty) = 0$.)

QUESTION 3 [Counts 15%.]

A one-dimensional *anharmonic* oscillator may be described with the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \alpha x^3 + \beta x \dot{x}^2$$

where the mass of the oscillator equals 1.

a) What is the Hamiltonian $H(x, p)$ of the anharmonic oscillator?

A one-dimensional *harmonic* oscillator may be described with the Lagrangian

$$L_0 = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$$

and the Hamiltonian

$$H_0 = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2.$$

b) Write

$$L = L_0 + \delta L$$

and

$$H = H_0 + \delta H$$

for the anharmonic oscillator, and show that for small oscillations (i.e., $|\alpha x| \ll \omega^2$ and $|\beta x| \ll 1$), the "extra terms" in L and H are equal, but with opposite signs, i.e.,

$$\delta L = -\delta H.$$

QUESTION 4 [For physics students, counts 25%.]

A possible Lagrangian for a relativistic particle with mass m, moving in a conservative potential $V(x_i)$, is

$$L = -\frac{mc^2}{\gamma} - V$$

a) Show that this Lagrangian results in Lagrange equations corresponding to a generalized version of Newton's 2. law,

$$\frac{d}{dt}\,p_i = F_i$$

Here, p_i are spatial components ($i = 1, 2, 3$) of the four-momentum p_μ.

b) Show that the Hamiltonian

$$H = v_i p_i - L$$

then equals the total energy of the particle.

Let the particle be an electron with mass $m = m_e$ and charge $q = -e$, starting at the origin with zero velocity at time $t = 0$, and let the potential represent a uniform electric field pointing along $-\hat{x}$, i.e.,

$$V(x) = qE_0 x$$

c) Write down the Lagrange equation for the electron and show that its speed becomes

$$\dot{x} = \frac{eE_0 t/m_e}{\sqrt{1 + (eE_0 t/m_e c)^2}}$$

d) Solve this equation and find the path $x(t)$ of the electron. Check that your answer makes sense, both for small and for large values of t.

QUESTION 5 [For cybernetics students, counts 25%.]

In Norwegian only.

a) Den aller enkleste differensligningen er gitt ved relasjonen

$$a_{n+1} = r\, a_n,$$

der $n \geq 0$ mens r er et reelt tall. Anta at verdien a_0 er kjent, og skriv ned uttrykket for den generelle a_n.

b) En representasjon av den lineære, homogene differensligningen av annen orden er

$$a_{n+1} = b\, a_n + c\, a_{n-1},$$

der $n \geq 1$ mens b og c er reelle tall. Sett som prøveløsning

$$a_n = C\, r^n,$$

og utled direkte fra differensligningen den karakteristiske annengradsligningen.

c) Anta at det er to distinkte røtter r_1 og r_2 for annengradsligningen i punkt *b*). Det oppgis at alle følger

$$a_n = c_1\, r_1^n + c_2\, r_2^n,$$

der c_1 og c_2 er konstanter, er løsninger av

$$a_{n+1} = b\, a_n + c\, a_{n-1}.$$

Sett opp et system for de to ukjente koeffisientene c_1 og c_2, og bestem eksplisitte uttrykk for c_1 og c_2, dvs uttrykt ved de kjente størrelsene r_1, r_2, a_0 og a_1. (HINT: Bruk gjerne Cramers regel ved løsning av systemet med de to ukjente.)

d) Betrakt følgen $\{F_n\}$ av Fibonaccitall, definert ved

$$F_{n+1} = F_n + F_{n-1},$$

der $F_0 = F_1 = 1$. Gitt at

$$F_n = C_1\, r_1^n + C_2\, r_2^n,$$

bestem de to distinkte røttene r_1 og r_2 ved å løse den tilhørende karakteristiske annengradsligningen. (HINT: Denne ligningen kan løses direkte ut fra analysen i punkt *b*).)

Some formulas

The validity of the formulas and the meaning of the symbols are assumed to be known.

- Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

- Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

- The Lorentz factor:

$$\gamma = 1/\sqrt{1-\beta^2} \quad , \quad \beta = v/c$$

- Four-vector:

$$x_\mu = (\boldsymbol{r}, ict)$$

- Proper time τ defined by:

$$dx_\mu \, dx_\mu = -c^2 \, d\tau^2$$

- Four-velocity:

$$u_\mu = \frac{dx_\mu}{d\tau} = \gamma(\boldsymbol{v}, ic)$$

- Four-momentum:

$$p_\mu = m \, u_\mu = \gamma(m\boldsymbol{v}, imc)$$

- Four-potential:

$$A_\mu = (\boldsymbol{A}, i\phi/c)$$

- Electromagnetic field:

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$
$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

- Lorentz transformation (with relative velocity $\boldsymbol{v} = v\hat{x}$):

$$L_{22} = L_{33} = 1 \;\; , \;\; L_{11} = L_{44} = \gamma \;\; , \;\; L_{14} = -L_{41} = i\beta\gamma$$

- Trigonometric relations:

$$\begin{aligned} \cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\ \sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \end{aligned}$$

- For $|x| \ll 1$ we have:

$$(1+x)^n \simeq 1 + nx \qquad \cos x \simeq 1 \qquad \sin x \simeq x$$